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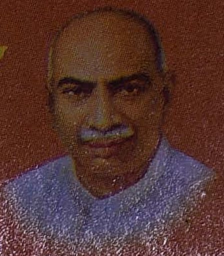

B.Sc., Mathematics

Third Year

PAPER - VI

REAL ANALYSIS AND COMPLEX ANALYSIS

UNIT : 6 - 10

Volume 2

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REAL ANALYSIS AND COMPLEX ANALYSIS

Dear Student,

We welcome you as a student of the third year B.Sc., degree course in Mathematics.This Paper-VI deals with REAL ANALYSIS AND COMPLEX ANALYSIS. The learning material for this paper will be supplemented by contact seminars.

Learning through the Distance Education mode, as you are all aware, involves self-learning and self-assessment and in this regard you are expected to put in disciplined and dedicated effort.On our part, we assume of our guidance and support.

Best wishes.

B.Sc., MATHEMATICS – THIRD YEAR

PAPER VI REAL ANALYSIS AND COMPLEX ANALYSIS

SYLLABUS

UNIT 1 : Sequences – Definition and examples – Convergent and divergent sequences – Cauchy sequences (definitions only) introduction of countable and uncountable sets- Holder's and Minkowski's inequalities – Metric space – Definition and examples.

UNIT 2 : Open sets and closed sets (definition and examples only) – Completeness-definition and examples – Cantor's intersection theorem and Baire's category theorem.

UNIT 3 : Contiunuity – Definition and Examples – Homemorphism (Discontinuous functions on R are not included).

UNIT 4 : Connected – Definition and examples – Connected subsets of R- connectedness and Continuity – Intermediate value theorem.

UNIT 5 : Compactness – Definition and examples – Compact subsets of R – Equivalent characterization of Compactness.

UNIT 6 : Analytic function -C.R.-equations – Sufficient conditions – Harmonic Functions.

UNIT 7 : Elementary Transformation – Bilinear Transformations – Cross ratio-fixed points - Special Bilinear Transformation – Real axis to real axis – Unit circle to unit circle and real axis to unit circle only.

UNIT 8 : Cauchy's Fundamental theorem – Cauchy's integral formulae and formulae of derivatives – Morera's theorem – Cauchy's inequality – Lioville's theorem – Fundamental theorem of algebra.

UNIT 9 : Taylor's theorem, Laurant's theorem – singular points – Poles – Argument principle – Rouche's theorem.

UNIT 10 : Calculus of Residues – Evaluation of Definite Integral.

Text Books: 1. Modern Analysis by S. Arumugam and A. Thangapandi Isaac. New Gamma Publishing house, 2005.

2. Complex Analysis by S. Arumugam , Thangapandi Isaac and A. Somasundaram, Sci. Tech Publications, Jan 2003.

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COMPLEX ANALYSIS

Space for Hints

0. INTRODUCTION

0.1 COMPLEX NUMBERS

Recall that the imaginary number i is defined to be $\sqrt{-1}$. We call the ***imaginary unit*** and construct any imaginary number by multiplying this unit with a real number. So we can represent all imaginary numbers in the form ***iy*** where y is real number.

A ***complex number*** is simply the sum of a real number and an imaginary number. We write

$z = x+iy,$

where z is a complex number and both x and y are real numbers. We can refer to the ***real*** and ***imaginary parts of*** z separately as

$Re\ z = x$ *and* $Im\ z = y.$

If $x=0$ $(y=0)$ we say that the complex number z is ***purely imaginary (real).***

0.1.1 BASIC OPERATIONS

To add (subtract) complex numbers, we simply add (subtract) the real and imaginary parts. For example if we have complex numbers

$z_1= x_1+iy_1,\ z_2= x_2+iy_2,$

then

$z_1+z_2 = (x_1+x_2) + i(y_1+y_2)$
$z_1-z_2 = (x_1-x_2) + i(y_1-y_2)$

We can multiply complex numbers using the distributive law and the fact that $i^2=-1$.

For z_1 and z_2 as above we have that

$z_1z_2 = (x_1x_2 - y_1y_2) + i\ (x_1y_2+y_1x_2).$

In the case of multiplying a complex number by a real scalar c, we see that this formula reduces to

$cz = c(x + iy) = cx+ciy.$

Space for Hints

In order to divide complex numbers, we first introduce the ***complex conjugate***. For a complex number z the complex conjugate $\bar{z}$ is given by

$$\bar{z} = x - iy.$$

In other words

$$Re\bar{z}=Re\, z \text{ and } Im\,\bar{z}= -Im\, z,$$

which leads to the identities

$$Re\, z = \frac{z+\bar{z}}{2} \text{ and } \mathrm{Im}\, z = \frac{z-\bar{z}}{2i}$$

Note that $z\bar{z}$ is always a real number. In particular

$$\bar{z}z = x^2+y^2.$$

Therefore we can find the ***inverse*** of a complex number $z\neq 0$ as

$$z^{-1} = \frac{1}{z\bar{z}}\bar{z} = \frac{1}{x^2+y^2}\bar{z}$$

Indeed,

$$z^{-1}z = \frac{1}{x^2+y^2}\bar{z}z = \frac{1}{x^2+y^2}(x^2+y^2) = 1$$

If we define as

$$\frac{z_1}{z_2} = z_1 z_2^{-1}\ (z_2 \neq 0),$$

then

$$\frac{z_1}{z_2} = \frac{z_1\bar{z}_2}{z_2\bar{z}_2} = \frac{(x_1x_2+y_1y_2)+i(y_1x_2-x_1y_2)}{x_2^2+y_2^2}$$

0.1.2 THE COMPLEX PLANE

In the previous section we treated a complex number simply as the sum of a real number and an imaginary number. The allowed us to do the basic arthimethic above, but it be more useful us going forward to treat complex numbers as vectors in a two-dimensional real vector space.

That's mouthful, but all we mean is that we represent each complex number as an ordered pair *(x,y)* of two real numbers. So for $z = (x,y) = x+iy$ we have that

$$Re\ z = x \text{ and } Im\ z = y.$$

We can represent our vectors graphically on the standard co-ordinate plane as directed arrows from the origin to the point *(x,y)* as in the first figure. We call this space the ***complex plane***. By our definition of the ordered pair *(x,y),* we see that purely real numbers will fall on the *x*-axis, or what we call the ***real axis (line).*** Similarly purely imaginary numbers will fall on the *y*-or ***imaginary axis.***

Now that we are working with vectors, it will be useful for us to have a concept of length. We can use the Pythagorean theorem on the complex plane to determine the length of any vector. For the complex number $z = (x,y)$ we see that its length $\sqrt{x^2+y^2}$. We call this the modulus of z and denote it by $|z|$. Note that

$$|z|^2 = x^2+y^2 = z\bar{z}.$$

Complex conjugation corresponds to reflecting a vector across the real axis as shown in the first figure. As we expect, complex conjugation does not change the modulus of a complex number. In fact

$$|\bar{z}|^2 = \bar{z}\bar{\bar{z}} = \bar{z}z = z\bar{z} = |z|^2,$$

where we use the fact $\bar{\bar{z}} = z$.

On the complex plane vector addition using the parallelogram rule, as shown in the second Figure, corresponds to addition of complex numbers. Multiplying a complex number z by a real scalar c corresponds to lengthening the vector z by a factor of c. Indeed,

$$|cz| = \sqrt{(cx)^2+(cy)^2} = |c|\sqrt{x^2+y^2} = |c||z|.$$

Often we will refer to regions in the complex plane by using an equqtion. For example $|z|=1$ specifies the unit circle : the set of points with modulus 1.

0.1.3. POLAR FORM

Recall from vector calculus that when working with two-dimensional vectors on the coordinate plane, we have a choice of coordinate systems to use. As you have surely seen, polar coordinates make the solutions of some problems easier. In the complex plane we

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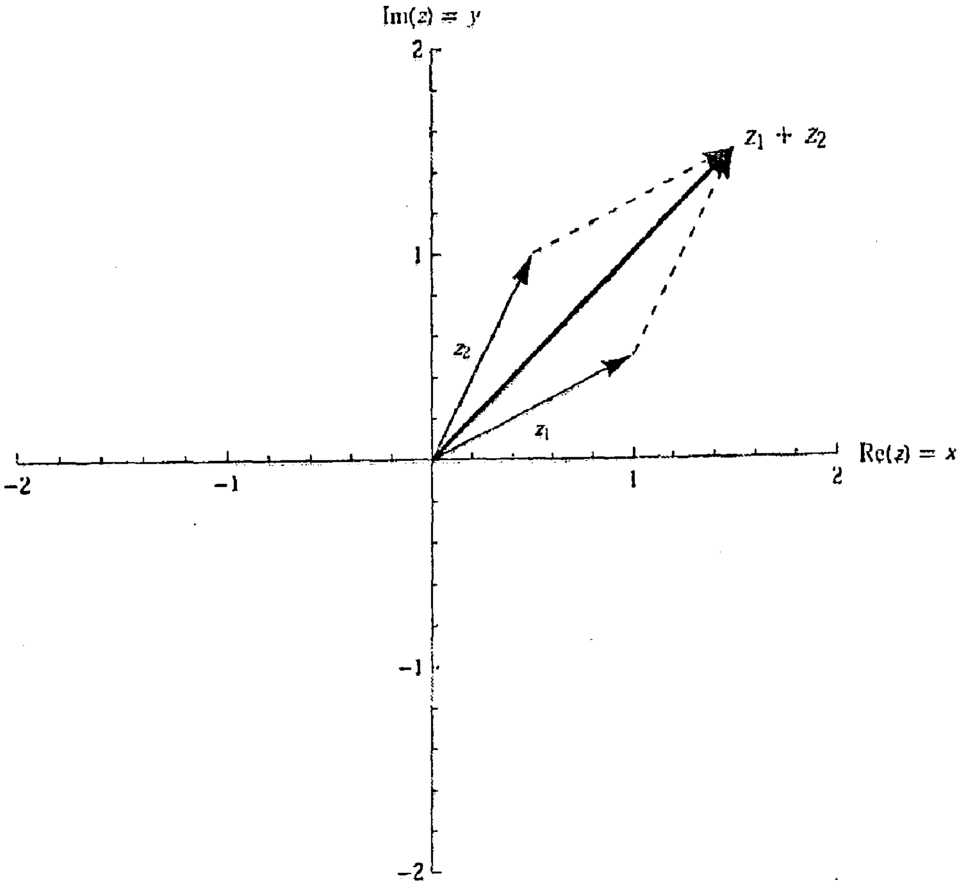
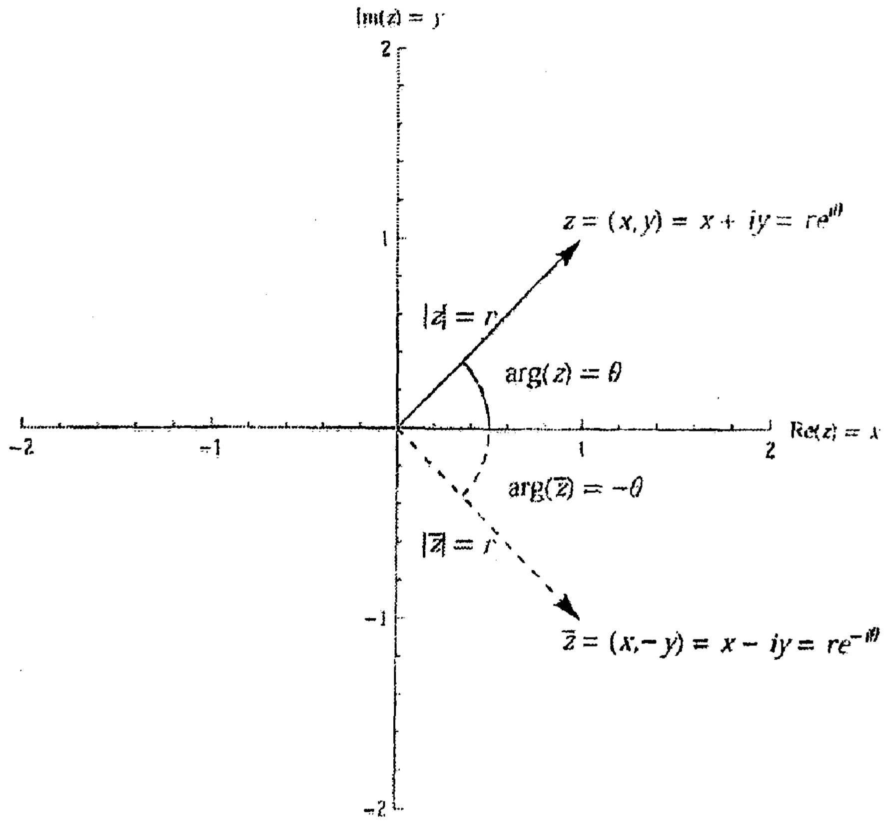
have an equivalent notion of polar coordinates, and it will lead us to one of the most celebrated formulas in mathematics.



A complex number $z = x+iy$ is shown as a vector in the complex plane and in polar form as $z = re^i\theta$. Its complex conjugate $\bar{z}$ is its reflection across the real axis.



Adding two complex numbers z_1 and z_2 in the complex plane using the parallelogram rule

Space for Hints

Instead of referring to a complex number $z = x+iy$ by its real and imaginary parts (rectangular form), we could instead to it by its modulus $|z|$ and the angle it makes with the positive reals axis (measured counter clockwise). We call this angle the ***argument*** of z and denote it by arg z. Look again at the first Fgure. Note that the choice of the argument is not unique. The angel θ is equivalent to the angle $\theta + 2k\pi$ for any integer k. Therefore ***arg z*** is really an infinite set of angles.

To obtain a unique for the argument we will restrict ourselves to choosing angles in the interval $(0, 2\pi)$. This restriction make the arg function single – valued, and we call this the ***principal value*** of the argument. We denote it as Arg (with a capital 'A'). The set argz can now be expressed as

$$arg\, z = \{Arg\, z + 2k\pi : k\epsilon z\}.$$

If we let

$$r = |z| = \sqrt{x^2 + y^2} \text{ and } \theta = Arg\, z = tan^{-1} \frac{y}{x}, 4$$

we see that

$$x = Re\, z = r\, cos\theta$$
$$y = Im\, z = r\, sin\theta$$

Therefore

$$z = x+iy = r(cos\theta + i\, sin\theta)$$

This representation of a complex number does not seem immediately useful. However Euler's formula tells us that

$$e^i\theta = cos\theta + i\, sin\, \theta.$$

From this, we get the polar form of a complex number.

$$z = re^i\theta,$$

where r and θ are the modulus and argument of z respectively.

It also leads to Euler's identity, the most beautiful equation in mathematics.

$$e^{i\pi} + 1 = 0$$

Space for Hints

This single equation links the three basic arithmetic operations(addition, multiplication and exponentiation), the additive (0), the multiplicative identity (1) and three fundamental constants (i, π and e). How do we see that it is true? If we take $e^{i\pi}$ as a complex number in polar form, we see that it has modulus 1 and argument π. We can plot it (on paper or mentally) or use the formulas above to find that

$$\text{Re } e^{i\pi} = 1.\cos\pi = -1$$
$$\text{Im } e^{i\pi} = 1.\sin\pi = 0.$$

So $e^{i\pi} = -1$.

Space for Hints

UNIT 6

6.1 ANALYTIC FUNCTIONS

Introduction

We can represent any complex number $x+iy$ by a point (x,y) in $\mathbf{R} \times \mathbf{R}$. The plane $\mathbf{R} \times \mathbf{R}$ representing the complex numbers in this way is called the ***complex plane***. Let $S \subseteq \mathbf{C}$. Then S is called a ***connected set*** if every pair of points in S can be joined by a polygon which lies in S. Any nonempty open connected subset of C is called a ***region*** in **C**.

Definition 6.1.1

A function f defined in a region D of the complex plane is said to be ***analytic at a point*** $a \in D$ if f is differential at every point of some neighbourhood of a.

Thus f is analytic at a if there exists $\varepsilon > 0$ such that f is differentiable at every point of the disc $S(a,\varepsilon) = \{z \,/\, |z - a| < \varepsilon\}$.

If f is analytic at every point of a region D, then f is said to be analytic in D.

A function which is analytic at every point of the complex plane is called an ***entire function*** or ***integral function.*** For example any polynomial is an entire function.

Remark 6.1.2

If $f(z)$ is analytic at a point a , then there exists $\varepsilon > 0$ such that $f(z)$ is differentiable at each point of $S(a, \varepsilon)$. Now, let $z \in S(a, \varepsilon)$. Then we can find $\delta > 0$ such that $S(z, \delta) \subseteq S(a, \varepsilon)$. Hence f is differentiable at every point of $S(z, \delta)$ so that f is analytic at z.

Thus f is analytic at every point of $S(a, \varepsilon)$. Hence f is analytic at a and if only if f is analytic at each point of some neighbourhood of a. Hence the set of all points for which a given function is analytic forms an open set.

In particular, if a function is analytic in an arbitrary subset A of the complex plane, then there exists an open set containing A in which the function is analytic.

Remark 6.1.3

$$f'(z) = u_x(x,y) + iv_x(x,y) = v_y(x,y) - iu_y(x,y)$$

is further differentiable and hence $f'(z)$ is continuous.
Hence u_x, v_x, u_y, v_y are all continuous.

Further it follows that if $f(z)$ is analytic in D, then u and v have continuous partial derivatives of all orders.

Space for Hints

Theorem 6.1.4
An analytic function in a region D with its derivative zero at every point of the domain is a constant.

Proof.
Let $f(z) = u(x,y)+iv(x,y)$ be analytic in D and $f'(z) = 0$ for all $z \in D$.
Since $f'(z) = u_x + iv_x = v_y - iu_y$, we have $u_x = u_y = v_x = v_y = 0$.
Therefore $u(x,y)$ and $v(x,y)$ are constant functions and hence $f(z)$ is constant.

Remark 6.1.5
The above theorem is not true if the domain of $f(z)$ is not a region.
For example, let $D = \{z / |z| < 1\} \cup \{z / |z| > 2\}$
D is not a connected subset of **C** so that D is not a region.
Let $f: D \rightarrow$ **C** be defined by

$$f(z) = \begin{cases} 1 & \text{if } |z| < 1 \\ 2 & \text{if } |z| > 2 \end{cases}$$

Clearly $f'(z) = 0$ for all points $z \in D$ and f is not a constant function in D.

Solved Problems

Problem 6.1.6
An analytic function in a region with constant modulus is constant.

Solution.
Let $f(z) = u(x,y)+iv(x,y)$ be analytic in a domain D.
Since $|f(z)|$ is constant, we have $u^2+v^2 = c$ where c is a constant.
Differentiating partially with respect to x, we get $2uu_x + 2vv_x = 0$.

(i.e) $uu_x + vv_x = 0$. ....(1)

Similarly differentiating partially with respect to y, we get

$uu_y + vv_y = 0$. ....(2)

Using C-R equations in (1) and (2) ,we get

$uu_x - vu_y = 0$. ....(3)

$uu_y + vu_x = 0$. ....(4)

Eliminating u_y from (3) and (4), we get $(u^2+v^2)\, u_x = 0$.
Since $u^2+v^2 = c$, we get $u_x = 0$.
Similarly we can prove that $v_x = 0$ so that $f'(z) = u_x + iv_x = 0$.
Hence f is constant.

Problem 6.1.7
Any analytic function $f(z) = u + iv$ with $\arg f(z)$ constant is itself a constant function.

Solution.
We know that $\arg f(z) = \tan^{-1}(v/u) = c$, where c is a constant.

Space for Hints

Therefore $v/u = k$ where k is a constant.
Therefore $v = ku$.
Hence $v_x = ku_x$ and $v_y = ku_y$.
Eiminating k from the above equations , we get $u_x v_y = v_x u_y$.
Therefore $u_x v_y - u_y v_x = 0$.
Therefore $u_x^2 + u_y^2 = 0$ (using C- R equations given in 6.2)
Therefore $u_x = 0$ and $u_y = 0$ and hence u is constant.
Similarly we can prove that v is constant.
Therefore $f = u+iv$ is a constant.

Problem 6.1.8

If $f(z)$ and $\overline{f(z)}$ are analytic in a region D, show that $f(z)$ is a constant in that region.

Solution.

Let $f(z) = u(x,y) + iv(x,y)$.

Therefore $\overline{f(z)} = u(x,y) - iv(x,y)$.
$= u(x,y) + i[-v(x,y)]$.

Since $f(z)$ is analytic in D, we have $u_x = v_y$ and $u_y = -v_x$.
(By C-R equations)

Since $\overline{f(z)}$ is analytic in D, we have, $u_x = -v_y$ and $u_y = v_x$.
Adding ,we get $u_x = 0$ and $u_y = 0$.
Hence $u_x = 0 = v_x$.
Therefore $f'(z) = u_x + iv_x = 0$
Therefore $f(z)$ is constant in D.

Problem 6.1.9

Prove that the functions $f(z)$ and $\overline{f(\bar{z})}$ are simultaneously analytic.

Solution.

Suppose $f(z) = u(x,y) + iv(x,y)$ is analytic in a region D.
Then the first order partial derivatives of u and v are continuous and satisfy the C-R equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{.........(1)}$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \text{..........(2)}$$

Now, $\overline{f(\bar{z})} = u(x,-y) - iv(x,-y)$.
$= u_1(x,y) + iv_1(x,y)$ where $u_1(x,y) = u(x,-y)$ and $v_1(x,y) = -v(x,-y)$.

$$\frac{\partial u_1}{\partial x} = \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \frac{\partial v_1}{\partial y} \quad from(1)$$

$$\frac{\partial u_1}{\partial y} = -\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} = -\frac{\partial v_1}{\partial x}$$

The first order partial derivatives of u_1 and v_1 are continuous and satisfy the Cauchy-Riemann equations in D.

Hence $\overline{f(\overline{z})}$ is analytic in D.

Similarly if $\overline{f(\overline{z})}$ is analytic in D, then $f(z)$ is also analytic in D.

Problem 6.1.10

If $\dfrac{\partial^2}{\partial x\partial y} = \dfrac{\partial^2}{\partial y\partial x}$, prove that $\dfrac{\partial^2}{\partial x^2} = \dfrac{\partial^2}{\partial y^2} = 4\dfrac{\partial^2}{\partial z\partial \overline{z}}$

Solution.

$$Let\ z = x+iy.$$

$$\therefore x = \frac{1}{2}(z+\overline{z}) \quad \text{and} \quad y = \frac{1}{2i}(z-\overline{z})$$

Hence

$$\frac{\partial}{\partial \overline{z}} = \frac{\partial}{\partial x}\frac{\partial x}{\partial \overline{z}} + \frac{\partial}{\partial y}\frac{\partial y}{\partial \overline{z}}$$

$$= \frac{1}{2}\frac{\partial}{\partial x} - \frac{1}{2i}\frac{\partial}{\partial y}$$

$$= \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

$$\therefore \frac{\partial^2}{\partial z\partial \overline{z}} = \frac{1}{2}\left[\left(\frac{\partial^2}{\partial x^2} + i\frac{\partial^2}{\partial x\partial y}\right)\left(\frac{1}{2}\right) + \left(\frac{\partial^2}{\partial y\partial x} + i\frac{\partial^2}{\partial y^2}\right)\left(\frac{1}{2i}\right)\right]$$

$$= \frac{1}{4}\left[\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + i\frac{\partial^2}{\partial x\partial y} + \frac{1}{i}\frac{\partial^2}{\partial y\partial x}\right]$$

$$= \frac{1}{4}\left[\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + \frac{\partial^2}{\partial x\partial y}\left(i + \frac{1}{i}\right)\right]$$

$$= \frac{1}{4}\left[\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\right]$$

$$\therefore \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z\partial \overline{z}}.$$

Space for Hints

Exercises 6.1.11

1. Prove that an analytic function whose real part is constant is itself a constant.
2. Prove that an analytic function whose imaginary part is constant is itself a constant.
3. If $f = u + iv$ is analytic in a region D and uv is constant in D, then prove that f reduces to a constant.
4. If $f = u+iv$ is analytic in a region D and $v = u^2$ in D, then prove that f reduces to a constant.
5. Determine the constants a and b in order that the function $f(z)= (x^2 + ay^2 - 2xy) + i(bx^2 - y^2 + 2xy)$ should be analytic. Find $f'(z)$.

Answers: 5. $a=-1; b=1; f'(z)=(1+i)z^2$

Space for Hints

6.2 THE CAUCHY – RIEMANN EQUATIONS

The existence of the derivative of complex function of a complex variable $f(z)$ requires $\dfrac{f(z+h)-f(z)}{h}$ to approach to the same limit as $h \to 0$ along any path. This has some far reaching consequences. In this section we derive some important properties of the real and maginary parts of the differentiable function $f(z) = u(x,y) + i\,v(x,y)$.

Therorem 6.2.1

Let $f(z) = u(x,y) + i\,v(x,y)$ be differentiable at a point $z_0 = x_0 + i\,y_0$ Then $u(x,y)$ and $v(x,y)$ have first order partial derivative $u_x(x_0,y_0)$, $u_y(x_0,y_0)$, $v_x(x_0,y_0)$ and $v_y(x_0,y_0)$ at (x_0,y_0) and these partial derivatives satisfy the **Cauchy – Riemann equations (C-R equations)** given by $u_x(x_0,y_0) = v_y(x_0,y_0)$ and $u_y(x_0,y_0) = - v_x(x_0,y_0)$.

Also, $f'(z_0) = u_x(x_0,y_0) + i\,v_x(x_0,y_0)$.
$= v_y(x_0,y_0) - i u_y(x_0,y_0)$.

Proof.

Since $f(z) = u(x,y) + i\,v(x,y)$ is differentiable at $z_0 = x_0 + i\,y_0$,

$\lim\limits_{h\to 0}\dfrac{f(z_0+h)-f(z_0)}{h}$ exists and hence the limit is independent of the path in which h approaches zero.

Let $h = h_1 + ih_2$.

Now $\dfrac{f(z_0+h)-f(z_0)}{h}$

$$= \frac{u(x_0+h_1, y_0+h_2) + i\,v(x_0+h_1, y_0+h_2) - u(x_0,y_0) - i\,v(x_0,y_0)}{h_1 + i\,h_2}$$

$$= \left[\frac{u(x_0+h_1, y_0+h_2) - u(x_0,y_0)}{h_1 + i\,h_2}\right] + i\left[\frac{v(x_0+h_1, y_0+h_2) - v(x_0,y_0)}{h_1 + ih_2}\right]$$

Suppose $h \to 0$ along the real axis so that $h = h_1$. Then

Space for Hints

$$f'(z_0) = \lim_{h_1 \to 0} \frac{f(z_0 + h_1) - f(z_0)}{h_1}$$

$$= \lim_{h_1 \to 0} [\frac{u(x_0 + h_1, y_0 + h_2) - u(x_0, y_0)}{h_1}]$$

$$+ i \lim_{h_1 \to 0} [\frac{v(x_0 + h_1, y_0 + h_2) - v(x_0, y_0)}{h_1}]$$

$= u_x(x_0,y_0) + i\, v_x(x_0,y_0). \rightarrow (1)$

Now, suppose $h \rightarrow 0$ along the imaginary axis so that $h = ih_2$.

$$f'(z_0) = \lim_{ih_2 \to 0} \frac{f(z_0 + h_2) - f(z_0)}{i\, h_2}$$

$$= \lim_{h_2 \to 0} [\frac{u(x_0, y_0 + h_2) - u(x_0, y_0)}{ih_2}]$$

$$+ i \lim_{h_2 \to 0} [\frac{v(x_0, y_0 + h_2) - v(x_0, y_0)}{ih_2}]$$

$$= [\frac{u_y(x_0, y_0)}{i}] + i\, [\frac{v_y(x_0, y_0)}{i}]$$

$$= \frac{1}{i}\, u_y(x_0,y_0) + v_y(x_0,y_0)$$

$= -\, iu_y(x_0,y_0) + v_y(x_0,y_0)\, . \quad \rightarrow (2)$

From (1) and (2) we get

$f'(z_0) = u_x(x_0,y_0) + i\, v_x(x_0,y_0). = v_y(x_0,y_0) - iu_y(x_0,y_0).$

Equating real and imaginary parts, we get

$$u_x(x_0,y_0) = v_y(x_0,y_0)$$
$$u_y(x_0,y_0) = -\, v_x(x_0,y_0).$$

Remark 6.2.2

Since $f'(z) = u_x + i\, v_x = v_y - iu_y$, we have

$|f'(z)|^2 = u_x^2 + v_x^2 = v_y^2 + u_y^2.$

Also $|f'(z)|^2 = u_x^2 + u_y^2 = v_x^2 + v_y^2.$

Further $|f'(z)|^2 = u_x v_y - u_y v_x$

$$= \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$$

$$= \frac{\partial(u,v)}{\partial(x,y)}$$

Remark 6.2.3

The Cauchy-Riemann equations provide a necessary condition for differentiability at a point. Hence if the C-R equations are not satisfied for a complex function at a point ,then we can conclude

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that the function is not differentiable.

For example, consider the function

$$f(z) = \bar{z} = x - iy.$$

Here $u(x,y) = x$ and $v(x,y) = -y$.

Therefore $u_x(x,y) = 1$ and $v_y(x,y) = -1$.

Therefore $u_x \neq v_y$ so that C-R equations are not satisfied at any point z.

Hence the function $f(z) = \bar{z}$ is nowhere differentiable.

Remark 6.2.4

The C-R equations are not sufficient for differentiability at a point as shown in the following examples.

Example 6.2.5

$$\text{Let } f(z) = \begin{cases} \dfrac{xy}{x^2+y^2} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$$

$$\text{Here } u(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

And $v(x,y) = 0$

$$\text{Now, } u_x(0,0) = \lim_{h\to 0} \frac{u(h,0)-u(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{0-0}{h} = 0.$$

Similarly, $u_y(0,0) = 0$.

Also $v_x(0,0) = 0$ *and* $v_y(0,0) = 0$.

Hence C-R equations are satisfied at $z = 0$.

Now , along the path $y = mx$,

$$f(z) = \frac{xmx}{x^2+m^2x^2} = \frac{m}{1+m^2} \quad \text{if } x \neq 0.$$

Hence if $z \to 0$ along the path $y = mx$, $f(z) \to \dfrac{m}{1+m^2}$ which is different for different values of m.

Hence $f(z)$ does not have a limit as $z \to 0$ so that $f(z)$ is not even continuous at $z = 0$.

Thus $f(z)$ is not differentiable at $x = 0$.

Example 6.2.6

Let $f(z) = \sqrt{|xy|}$

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Here $u(x,y) = \sqrt{|xy|}$ and $v(x,y) = 0$.

$$u_x(0,0) = \lim_{h\to 0}\frac{u(h,0)-u(0,0)}{h}$$
$$= 0$$

Similarly, $u_y(0,0) = 0$.

Also $v_x(0,0) = 0$ and $v_y(0,0) = 0$.

Hence C-R equations are satisfied at $z = 0$.

We claim that $f(z)$ is not differentiable at $(0,0)$.

Along the path $y = mx$,

$$\frac{f(z)-f(0)}{z} = \frac{\sqrt{|xmx|}}{x+imx} = \frac{\sqrt{|m|}}{1+im} \quad \text{if } x \neq 0.$$

Hence as $z \to 0$ along the path $y = mx$,

$\dfrac{f(z)-f(0)}{z}$ *tends to* $\dfrac{\sqrt{|m|}}{1+im}$ which depends on the path along which $z \to 0$ so that f is not differentiable at $z = 0$.

Note 6.2.7

In this example, the function $f(z)$ is continuous and has partial derivatives which satisfy Cauchy–Riemann equations at 0 but is not differentiable at 0.

In the following theorem, we prove that C-R equations together with the continuity of partial derivaties give a sufficient condition for differentiability of complex function.

Theorem 6.2.8

Let $f(z) = u(x,y) + i\, v(x,y)$ be a function defined in a region D such that u, v and their first order partial derivatives are continuous in D. If the first order partial derivatives of u, v satisfy the Cauchy – Riemann equations at a point $(x,y) \in D$, then f is differentiable at $z = x+iy$.

Proof.

Since $u(x,y)$ and its first order partial derivatives are continuous at (x,y), we have by the mean value theorem for functions of two variables,

$u(x+h_1, y+h_2) - u(x,y) = h_1 u_x(x,y) + h_2 u_y(x,y) + h_1\epsilon_1 + h_2\epsilon_2$ ....(1)

where ε_1 and $\varepsilon_2 \to 0$ as h_1 and $h_2 \to 0$.

Similarly,

$v(x+h_1, y+h_2) - v(x,y) = h_1 v_x(x,y) + h_2 v_y(x,y) + h_1\epsilon_3 + h_2\epsilon_4$ ....(2)

where ε_3 and $\varepsilon_4 \to 0$ as h_1 and $h_2 \to 0$.

Let $h = h_1 + ih_2$.

Then $\dfrac{f(z+h)-f(z)}{h} = \dfrac{1}{h}[\, u(x+h_1, y+h_2) - u(x,y) + i \quad v(x+h_1, y+h_2) - v(x,y)]$

$= \frac{1}{h}[\ \{h_1\ u_x(x, y) + h_2\ u_y(x, y) + h_1\varepsilon_1 + h_2\varepsilon_2\} + i\ \{h_1$
$v_x(x, y) + h_2\ v_y(x, y) +$

$h_1\varepsilon_3 + h_2\varepsilon_4\}]$

using (1) and (2).

$= \frac{1}{h}[\ h_1\{\ u_x(x, y) + i\ v_x(x, y)\ \} + \ h_2\ \{u_y(x, y) +$
$iv_y(x, y)\}\ + h_1(\varepsilon_1 + i\ \varepsilon_3)$

$+ h_2(\varepsilon_2 + i\ \varepsilon_4)].$

$= \frac{1}{h}[\ (h_1 + \ ih_2)\ u_x\ (x,y) - i(h_1 + \ ih_2)\ u_y(x,y) +$
$h_1(\varepsilon_1 + i\ \varepsilon_3) + h_2(\varepsilon_2 + i\ \varepsilon_4)].$

(using C-R equations).

$= \frac{1}{h}[\ h\ u_x\ (x,y) - ih\ u_y(x,y) + h_1(\varepsilon_1 + i\ \varepsilon_3) + h_2(\varepsilon_2 + i$
$\varepsilon_4)].$

$= u_x\ (x,y) - i\ u_y(x,y) + \frac{h_1}{h}(\varepsilon_1 + i\ \varepsilon_3) + \frac{h_2}{h}\ (\varepsilon_2 + i\ \varepsilon_4).$

Now, since $|\frac{h_1}{h}| \le 1,\ \frac{h_1}{h}(\varepsilon_1 + i\ \varepsilon_3) \to 0$ as $h \to 0.$

Similarly $\frac{h_2}{h}\ (\varepsilon_2 + i\ \varepsilon_4) \to 0$ as $h \to 0.$

Therefore $\lim_{h\to 0} \frac{f(z+h)-f(z)}{h} = u_x\ (x,y)\ - i\ u_y(x,y)$. Hence f is

differentiable.

Example 6.2.9

Let $f(z) = e^x(\cos y + i \sin y)$.

Therefore $u(x,y) = e^x \cos y$ and $v(x,y) = e^x \sin y$.

Then $u_x(x,y) = e^x \cos y = v_y(x,y)$ and $u_y(x,y) = -e^x \sin y = -v_x(x,y)$.

Thus the first order partial derivatives of u and v satisfy the

Cauchy-Riemann equations at every point. Further $u(x,y)$ and $v(x,y)$

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and their first order partial derivatives are continuous at every point.

Hence f is differentiable at every point of the complex plane.

Example 6.2.10

Let $f(z) = |z|^2$.

Therefore $f(z) = u(x,y) + iv(x,y) = x^2+y^2$.

Therefore $u(x,y) = x^2+y^2$ and $v(x,y) = 0$.

Hence $u_x(x,y) = 2x$; $u_y(x,y) = 2y$; $v_x(x,y) = 2x = v_y(x,y)$.

Clearly the Cauchy-Riemann equations are satisfied at $z = 0$.

Further u and v and their first order partial derivatives are continuous and hence f is differentiable at z .

Also we notice that the C-R equations are not satisfied at any point $z \neq 0$ and hence f is not differentiable at $z \neq 0$.

Thus f is differentiable only at $z = 0$.

ALTERNATE FORMS OF CAUCHY-RIEMANN EQUATIONS

In the following theorem, we express the Cauchy-Riemann equations in complex form.

Theorem 6.2.11 (Complex form of C-R equations)

Let $f(z) = u(x,y) + iv(x,y)$ be differentiable. Then the C-R equations can be put in the complex form as $f_x = -if_y$.

Proof.

Let $f(z) = u(x,y) + iv(x,y)$.

Then $f_x = u_x + iv_x$ and $f_y = u_y + iv_y$.

Hence $f_x = -if_y$.

$\Leftrightarrow u_x + iv_x = -i(u_y + iv_y).$

$\Leftrightarrow$ $u_x + iv_x = v_y - iu_y$.

$\Leftrightarrow$ $u_x = v_y$ and $v_x = - u_y$.

Thus the two C-R equations are equivalent to the equation $f_x = -if_y$.

In the following theorem, we express the Cauchy Riemann equations and

the derivative of a

complex function in terms of it polar coordinates.

Theorem 6.2.12 (C.R Equations in Polar Coordinates)

Let $f(z) = u(r, \theta) + iv(r, \theta)$ be differentiable at $z = re^{i\theta} \neq 0$.

Then $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$ and $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$.

Further $f'(z) = \frac{r}{z}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)$.

Proof.

We have $x = r\cos\theta$ and $y = r\sin\theta$.

Hence $\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}$.

$$= \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta. \qquad \text{..........(1)}$$

Also $\frac{\partial v}{\partial \vartheta} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial \theta}$.

$$= \frac{\partial v}{\partial x}(-r\sin\theta) + \frac{\partial v}{\partial y}(r\cos\theta).$$

$$\therefore \frac{1}{r}\frac{\partial v}{\partial \theta} = -\frac{\partial v}{\partial x}\sin\theta + \frac{\partial v}{\partial y}\cos\theta.$$

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$$= \frac{\partial u}{\partial y}\sin\theta + \frac{\partial u}{\partial x}\cos\theta. \qquad \text{(using C-R equations)}$$

$$= \frac{\partial u}{\partial r} \qquad \text{(using (1))}$$

Thus $\dfrac{\partial u}{\partial r} = \dfrac{1}{r}\dfrac{\partial v}{\partial \theta}$.

Similarly we can prove that $\dfrac{\partial v}{\partial r} = -\dfrac{1}{r}\dfrac{\partial u}{\partial \theta}$.

Now,

$$r\left(\frac{\partial u}{\partial r} + \frac{\partial v}{\partial r}\right) = r\left[\left(\frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}\right) + i\left(\frac{\partial v}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial r}\right)\right]$$

$$= r\left[\left(\frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta\right) + i\left(\frac{\partial v}{\partial x}\cos\theta + \frac{\partial v}{\partial y}\sin\theta\right)\right]$$

$$= r\cos\theta\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) + r\sin\theta\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right).$$

$$= x\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) + iy\left(\frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}\right).$$

$$= x f'(z) + iy f'(z).$$

$$= (x+iy)\ f'(z).$$

$$= z f'(z).$$

$$\therefore f'(z) = \frac{r}{z}\left(\frac{\partial u}{\partial r} + \frac{\partial v}{\partial r}\right).$$

We now proceed to express C-R equations in yet another form.

Let $f(z) = u(x,y) + iv(x,y)$.

Since $x = \frac{z+\bar{z}}{2}$ and $y = \frac{z-\bar{z}}{2i}$ we have

$$f(z) = u\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right) + iv\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right).$$

Thus f can be thought of as a function of z and $\bar{z}$ are not independent variables we form the partial derivatives $\frac{\partial f}{\partial z}$ and $\frac{\partial f}{\partial \bar{z}}$ as is z and $\bar{z}$ are independent variables. With this convention we have the following theorem.

Theorem 6.2.13

If $f(z)$ is a differentiable function, the C-R equations can be put in the form $\frac{\partial f}{\partial \bar{z}} = 0.$

Proof.

$$\frac{\partial f}{\partial \bar{z}} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \bar{z}}.$$

$$= \frac{\partial f}{\partial x}\left(\frac{1}{2}\right) + \frac{\partial f}{\partial y}\left(-\frac{1}{2i}\right).$$

$$= \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right).$$

Thus $\frac{\partial f}{\partial \bar{z}} = 0 \Leftrightarrow \frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y}$ which is the complex form of the C-R equations.

Thus the C-R equations can be put in the form $\frac{\partial f}{\partial \bar{z}} = 0.$

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Solved problems

Problem 6.2.14

Verify Cauchy-Riemann equations for the function $f(z) = z^3$.

Solution.

$f(z) = z^3 = (x+iy)^3$.

$= (x^3 - 3xy^2) + i\,(3x^2y - y^3)$

$\therefore\ u(x,y) = x^3 - 3xy^2$ and $v(x,y) = 3x^2y - y^3$

$\therefore\ u_x = 3x^2 - 3y^2$ and $v_x = 6xy$.

$u_y = -6xy$ and $v_y = 3x^2 - 3y^2$.

Here $u_x = v_y$ and $u_y = -v_x$.

Hence the Cauchy-Riemann equations are statisfied.

Problem 6.2.15

Prove that the following functions are nowhere differentiable.

(i) $f(z) = Re\, z$. (ii) $f(z) = e^x (\cos y - i \sin y)$.

Solution.

(i) $f(z) = Re\, z = x$.

$\therefore\ u(x,y) = x$ and $v(x,y) = 0$.

$\therefore\ u_x = 1$ and $v_x = 0$.

$u_y = 0$ and $v_y = 0$.

Since $u_x \neq v_y$, the C-R equations are not satisfied at any point.

Hence $f(z)$ is nowhere differentiable.

(ii) $f(z) = e^x (\cos y - i \sin y)$.

$= e^x \cos y - i\, e^x \sin y$.

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$\therefore\ u(x,y) = e^x \cos y$ and $v(x,y) = -e^x \sin y$.

$\therefore\ u_x = e^x \cos y$ and $v_x = -e^x \sin y$.

$u_y = -e^x \sin y$ and $v_y = -e^x \cos y$.

Clearly C-R equations are not satisfied at any point and hence $f(z)$ is nowhere differentiable.

Problem 6.2.16

Prove that
$$f(z) = \begin{cases} \dfrac{z\,\text{Re}\,z}{|z|} & if\ z \neq 0 \\ 0 & if\ z = 0 \end{cases}$$

is continuous at $z = 0$ but not differentiable at $z = 0$.

Solution.

First we shall prove that $\lim_{z\to 0} f(z) = 0$.

Now $| f(z) - 0 | = \left|\dfrac{z\,\text{Re}\,z}{|z|}\right| = |\text{Re}\ z|$.

Further $|\,Re\ z\,| \le |z|$.

For any given $\varepsilon > 0$, if we choose $\delta = \varepsilon$, we get

$$|z| = |\,z - 0\,| < \delta \Rightarrow |\,f(z) - 0\,| < \varepsilon.$$

Hence f is continuous at $z = 0$.

Now, we prove that $f(z)$ is not differentiable at $z = 0$.

$$\frac{f(z) - f(0)}{z - 0} = \frac{z\,\text{Re}\,z}{z\,|z|} = \frac{\text{Re}\,z}{|z|}$$

$$= \frac{x}{\sqrt{x^2 + y^2}} \qquad where\ z = x + iy.$$

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Along the path $y = mx$,

$$\frac{f(z)-f(0)}{z-0} = \frac{x}{\sqrt{x^2+m^2x^2}} = \frac{1}{\sqrt{1+m^2}}$$

The value of the limit depends on m and hence on the path along which $z \to 0$.

$\therefore \lim_{z\to 0} \frac{f(z)-f(0)}{z-0}$ does not exist.

$\therefore f(z)$ is not differentiable at $z = 0$.

Problem 6.2.17

Prove that $f(z) = z\, Im\, z$ is differentiable only at $z = 0$ and find $f'(0)$.

Solution.

$f(z) = z\, Im\, z = (x+iy)y = xy + iy^2$

$\therefore u(x,y) = x\,y$ and $v(x,y) = y^2$.

$\therefore u_x = y;\ v_x = 0;\ u_y = x$ and $v_y = 2y$.

Clearly the C-R equation are satisfied only at $z = 0$.

Further all the first order partial derivatives are continuous.

Hence $f(z)$ is differentiable at $z = 0$.

Also $f'(0) = u_x(0,0) + iv_x(0,0) = 0$.

Problem 6.2.18

Show that $f(z) = \begin{cases} \dfrac{xy^2(x+iy)}{x^2+y^4} & if\ z \neq 0 \\ 0 \quad if \quad z = 0 \end{cases}$

is not differentiable at $z = 0$.

Solution.

$$\frac{f(z)-f(0)}{z-0}=\frac{xy^2(x+iy)}{x^2+y^4}\left(\frac{1}{x+iy}\right)$$

$$=\frac{xy^2}{x^2+y^4}$$

$\therefore$ Along the path $x = my^2$,

$$\frac{f(z)-f(0)}{z-0}=\frac{my^4}{m^2y^4+y^4}=\frac{m}{m^2+1}.$$

The value of the limit depends on m and hence depends on the path along which $z \to 0$.

$\therefore \lim\limits_{z\to 0} \dfrac{f(z)-f(0)}{z-0}$ does not exist.

$\therefore f(z)$ is not differentiable at $z = 0$.

Problem 6.2.19

Prove that the function $f(z) = \begin{cases} \dfrac{x^3(1+i)-y^3(1-i)}{x^2+y^2} & if\ z \neq 0 \\ 0 & if\ z=0 \end{cases}$ satisfies C-R equations at the origin but $f'(0)$ does not exist.

Solution.

$$f(z) = \begin{cases} \dfrac{x^3(1+i)-y^3(1-i)}{x^2+y^2} & if\ z \neq 0 \\ 0 & if\ z=0 \end{cases}$$

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Here $u(x,y) = \frac{x^3 - y^3}{x^2 + y^2}$ and $v(x,y) = \frac{x^3 + y^3}{x^2 + y^2}$ if $(x,y) \neq (0,0)$ and

$u(0,0) = v(0,0) = 0.$

Now, $u_x(0,0) = \lim_{h \to 0} \frac{u(h,0) - u(0,0)}{h}$

$$= \lim_{h \to 0} \left(\frac{h^3 / h^2 - 0}{h} \right) = 1.$$

Similarly $u_y(0,0) = -1$; $v_x(0,0) = 1$ and $v_y(0,0) = 1$.

Thus $u_x(0,0) = v_y(0,0) = 1$ and $u_y(0,0) = - v_x(0,0) = -1$, so that

C-R.equations are satisfied at z = 0.

Now, $$\frac{f(z) - f(0)}{z - 0} = \frac{x^3 - y^3}{(x^2 + y^2)(x + iy)} + i \frac{x^3 + y^3}{(x^2 + y^2)(x + iy)}$$

Along the path $y = mx$ we have

$$\frac{f(z) - f(0)}{z - 0} = \frac{x^3 - m^3 x^3}{(x^2 + m^2 x^2)(x + imx)} + i \frac{x^3 + m^3 x^3}{(x^2 + m^2 x^2)(x + imx)}$$

$$= \frac{1 - m^3}{(1 + m^2)(1 + im)} + i \frac{1 + m^3}{(1 + m^2)(1 + im)}$$

Hence the value of the limit depends on the path along which $z \to 0$.

Thus $\lim_{z \to 0} \frac{f(z) - f(0)}{z - 0}$ does not exist.

Hence f is not differentiable at 0.

Problem 6.2.20

Prove that $f(z) = \sin x \cosh y + i \cos x \sinh y$ is differentiable at every point.

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Solution.

$f(z) = \sin x \cosh y + i \cos x \sinh y.$

$\therefore u(x,y) = \sin x \cosh y$ and $v(x,y) = \cos x \sinh y$

$u_x = \cos x\ cushy$ and $v_x = -\sin x \sinh y$

$u_y = \sin x \sinh y$ and $v_y = \cos x \cosh y$

$\therefore u_x = v_y$ and $u_y = -v_x$ *for all x,y.*

Hence C-R. equations are satisfied at every point.

Further all the first order partial derivatives are continuous.

Hence $f(z)$ is differentiable at every point.

Problem 6.2.21

Find constants a and b so that the function $f(z) = a(x^2 - y^2) + ibxy + c$ is differentiable at every point.**Solution.**

Here $u(x,y) = a(x^2 - y^2) + c$ and $v(x,y) = bxy.$

$u_x = 2ax;\ v_x = by.$

$u_y = -2ay$ and $v_y = bx.$

Clearly $u_x = v_y$ and $u_y = -v_x.$

$\therefore$ The function $f(z)$ is differentiable for all values of a,b with $2a = b.$

Problem 6.2.22

Show that $f(z) = \sqrt{r}(\cos\theta/2 + i\sin\theta/2)$ where $r > 0$ and $0 < \theta < 2\pi$ is differentiable and find $f'(z)$.

Solution.

$f(z) = \sqrt{r}(\cos\theta/2 + i\sin\theta/2).$

$u = \sqrt{r}\cos(\theta/2)$ and $v = \sqrt{r}\sin(\theta/2).$

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$\therefore \dfrac{\partial u}{\partial r} = \dfrac{1}{2\sqrt{r}}\cos(\theta/2)$ and $\dfrac{\partial v}{\partial r} = \dfrac{1}{2\sqrt{r}}\sin(\theta/2)$

$\dfrac{\partial u}{\partial \theta} = \dfrac{-\sqrt{r}}{2}\sin(\theta/2)$ and $\dfrac{\partial v}{\partial \theta} = \dfrac{\sqrt{r}}{2}\cos(\theta/2)$

$$\text{Now, } \frac{1}{r}\frac{\partial v}{\partial \theta} = \frac{1}{r}\left(\frac{\sqrt{r}}{2}\cos(\theta/2) \right)$$

$$= \frac{1}{2\sqrt{r}}\cos(\theta/2)$$

$$= \frac{\partial u}{\partial r}.$$

$$\text{Thus } \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}.$$

$$\text{Similarly } \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}.$$

$$= \frac{1}{2\sqrt{r}}\sin(\theta/2).$$

Hence the C-R equations (in polar form) are satisfied.
Further all the first order partial derivatives are continuous.
Hence $f'(z)$ exists

$$\text{Also } f'(z) = \frac{r}{z}\left(\frac{\partial u}{\partial r} + \frac{\partial v}{\partial r}\right)$$

$$= \frac{r}{z}\left(\frac{1}{2\sqrt{r}}\cos(\theta/2) + \frac{i}{2\sqrt{r}}\sin(\theta/2)\right)$$

$$= \frac{r}{2\sqrt{r}z}\left(\cos(\theta/2) + i\sin(\theta/2)\right)$$

$$= \frac{1}{2z}\left[\sqrt{r}\left(\cos(\theta/2) + i\sin(\theta/2)\right)\right]$$

$$= \frac{1}{2z}\left[\sqrt{z}\right] = \frac{1}{2\sqrt{z}}.$$

$$\text{Hence } f'(z) = \frac{1}{2\sqrt{z}}.$$

Exercises 6.2.23

1. Verify C.R. equations for the following functions
   (i) $f(z) = az+b$.

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(ii) $f(z) = e^{\bar{z}}$.
(iii) $f(z) = (1/z), z \neq 0$.
(iv) $f(z) = iz+2$.
(v) $f(z) = e^{-x}(\cos y - i \sin y)$.
(vi) $f(z) = \cos x \cosh y - I \sin x \sinh y$.
(vii) $f(z) = \sin z$.
(viii) $f(z) = ze^{-z}$.

6.3 HARMONIC FUNCTIONS

Definition 6.3.1

Let $u(x,y)$ be a function of two real variables x and y defined in a region D. $u(x,y)$ is said to a ***harmonic function*** if $\partial^2u/\partial x^2 + \partial^2u/\partial y^2 = 0$ and this equation is called ***Laplace's equation***.

Theorem 6.3.2

The real and imaginary parts of an analytic function are harmonic functions.

Proof.

Let $f(z) = u(x,y) + iv(x,y)$ be an analytic function.
Then u and v have continuous partial derivatives of first order which satisfy the C-R equations given by $\partial u/\partial x = \partial v/\partial y$ and $\partial u/\partial y = -\partial v/\partial x$.
Further $\partial^2u/\partial x\partial y = \partial^2u/\partial y\partial x$ and $\partial^2v/\partial x\partial y = \partial^2v/\partial y\partial x$.
Now $\partial^2u/\partial x^2+\partial^2u/\partial y^2 = \partial/\partial x(\partial v/\partial y) + \partial/\partial y(-\partial v/\partial x)$
Thus u is a harmonic function.
Similarly we can prove that v is a harmonic function.

Remark 6.3.3

Laplace's equation provides a necessary condition for a function to be the real or imaginary part of an analytic function.

For example if $u(x,y) = x^2+y$, we have

$$\partial^2u/\partial x^2 = 2;\ \partial^2u/\partial y^2 = 0 \text{ and } \partial^2u/\partial x^2 + \partial^2u/\partial y^2 = 2.$$

Thus $u(x,y)$ is not harmonic function and hence it cannot be the real part of any analytic function.

Definition 6.3.4

Let $f = u+iv$ be an analytic function in a region D. Then v is said to be a ***conjugate harmonic*** function of u.

Theorem 6.3.5

Space for Hints

Let $f = u+iv$ be an analytic function in a region D. Then v is a harmonic conjugate of u if and only if u is a harmonic conjugate of $-v$.

Proof.

Let v be a harmonic conjugate of u. Then $f = u+iv$ is analytic.
Therefore $if = iu-v$ is also analytic.
Hence u is a harmonic conjugate of $-v$.
The proof for the converse is similar.

Theorem 6.3.6

Any two harmonic conjugates of a given harmonic function u in a region D differ by a real constant.

Proof.

. Let u be a harmonic function.
Let v and v^* be two harmonic conjugates of u.
$u + iv$ and $u + iv^*$ are analytic in D.
Hence by the Cauchy-Riemann equation, we have

$\partial u/\partial x = \partial v/\partial y = \partial v^*/\partial y$ and $\partial u/\partial y = -\partial v/\partial x = -\partial v^*/\partial x$

Therefore $\partial v/\partial y = \partial v^*/\partial y$ and $\partial v/\partial x = \partial v^*/\partial x$

Hence $\partial/\partial y(v - v^*) = 0$ and $\partial/\partial x(v - v^*) = 0$.

Therefore $v = v^* + c$ where c is a real constant.

Remark 6.3.7

The Cauchy-Riemann equations can be used to obtain a harmonic conjugate of a given harmonic function.
For example, let $u(x,y) = x^2 - y^2$.

Then $\partial^2u/\partial x^2 + \partial^2u/\partial y^2 = 2 - 2 = 0$ so that u is harmonic in the whole complex plane **C**.
Not, let $v(x,y)$ be a harmonic conjugate of u.

Then $\partial v/\partial y = \partial u/\partial x = 2x$ ....(1)

and $\partial v/\partial x = -\partial u/\partial y = -2y$ ....(2)

On integration of (1) with respect to y we get $v = 2xy+\varphi(x)$ where $\varphi(x)$ is a function of x alone.

Now from (2) $\partial v/\partial x = -\partial u/\partial y$ gives $2y + \varphi'(x) = 2y$.

Therefore $\varphi'(x) = 0$ so that $\varphi(x) = c$ (a constant)

Therefore $v = 2xy + c$.

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Thus the harmonic conjugate of $u(x,y) = x^2 - y^2$ is given by $v(x,y) = 2xy+c$ and the corresponding entire function is given by

$$f(z) = x^2 - y^2 + i(2xy+c)$$

$$= z^2 + ic.$$

Let $u(x,y)$ and $v(x,y)$ be given harmonic functions. We now describe a method, due to Milne-Thompson, of constructing an analytic function whose real part is $u(x,y)$ or imaginary part is $v(x,y)$.

Milne-Thompson method

Let $u(x,y)$ be a given harmonic function.
Let $f(z) = u(x,y) + iv(x,y)$ be an analytic function.

$$\text{Then } f'(z) = u_x(x,y) + iv_x(x,y)$$

$$= u_x(x,y) - iu_y(x,y).$$

Let $\varphi_1(x,y) = u_x(x,y)$ and $\varphi_2(x,y) = u_y(x,y)$.

We have $x = \dfrac{z+\bar{z}}{2}$ and $y = \dfrac{z-\bar{z}}{2i}$

$$\text{Hence } f'(z) = \varphi_1\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right) - i\varphi_2\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right).$$

Putting $z = \bar{z}$ we obtain $f'(z) = \varphi_1(z,0) - i\varphi_2(z,0)$.

Hence $f(z) = \int [\varphi_1(z,0) - i\,\varphi_2(z,0)]\, dz + c$.

Note that it can be proved in a similar way that the analytic function f(z) with a given harmonic function $v(x,y)$ as imaginary part is given by

$$f(z) = \int [\psi_1(z,0) - i\,\psi_2(z,0)]\, dz + c.$$

Where $\psi_1(x,y) = v_y$ and $\psi_2(x,y) = v_x$.

Solved Problems

Problem 6.3.8

Prove that $u = 2x - x^3 + 3xy^2$ is harmonic and find its harmonic conjugate. Also find the corresponding analytic function.

Space for Hints

Solution.

Given that $u = 2x - x^3 + 3xy^2$.

$u_x = 2 - 3x^2 + 3y^2$; $u_{xx} = -6x$; $u_y = 6xy$; $u_{yy} = 6x$.

$u_{xx} + u_{yy} = 0$. Hence u is harmonic.

Let v be a harmonic conjugate of u.

$f(z) = u+iv$ is analytic.
By Cauchy-Riemann equations, we have

$v_y = u_x = 2 - 3x^2 + 3y^2$.

Integrating with respect to y, we get

$$v = 2y - 3x^2y + y^3 + \lambda(x) \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

where $\lambda(x)$ is an arbitrary function of x.
$v_x = -6xy + \lambda'(x)$.

Now $v_x = -u_y$ gives $-6xy + \lambda'(x) = -6xy$.
Hence $\lambda'(x) = 0$ so that $\lambda(x) = c$ where c is a constant.
Thus $v = 2y - 3x^2y + y^3 + c$ [from (1)]

Now $f(z) = (2x - x^3 + 3xy^2) + i\,(2y - 3x^2y + y^3) + ic$

$= 2(x+i\,y) - [(x^3 - 3xy^2) + i(3x^2y - y^3)] + ic$

$= 2z - z^3 + ic$.

Therefore $f(z) = 2z - z^3 + ic$ is the required analytic function.

Problem 6.3.9

Show that $u = \log\sqrt{x^2 + y^2}$ is harmonic and determine its conjugate and hence find the corresponding analytic function $f(z)$.

Solution.

Let $u = \log\sqrt{x^2 + y^2} = \frac{1}{2}\log(x^2+y^2)$.

Therefore $u_x = x/(x^2+y)^2$;
$u_{xx} = ((x^2+y^2) - 2x)^2 / (x^2+y^2)^2$
$= (y^2 - x^2) / (x^2+y^2)^2$.

Similarly $u_{yy} = (x^2 - y^2) / (x^2+y^2)^2$.

Obviously $u_{xx}+u_{yy} = 0$ and hence u is harmonic.

Let v be a harmonic conjugate of u.

Therefore $f(z) = u+iv$ is an analytic function.

By C-R. equations, we have

$v_y= u_x = x/(x^2+y^2)$.

Integrating with respect to y, we get $v = tan^{-1}(y/x) + \phi(x)$ where $\phi(x)$ is an arbitrary function of x.

Now $v_x = [1/(1+y^2/x^2)]\,(-y/x^2)+ \phi'(x)$.

Also $v_x = -u_y \Rightarrow -y/[x^2+y^2] +\phi'(x) = -y/[x^2+y^2]$ so that $\phi'(x) = 0$.

Hence $\phi(x) = c$.

Therefore $v = tan^{-1}(y/x) + c$.

Therefore $f(z) = u + iv = log\sqrt{x^2+y^2} + i[tan^{-1}(y/x) + c]$.

Space for Hints

Problem 6.3.1

Show that $u(x,y) = sin\ x\ cosh\ y + 2\ cos\ x\ sinh\ y + x^2 - y^2 +4xy$ is harmonic. Find an analytic function $f(z)$ in terms of z with the given u for its real part.

Solution.

$u_x = cos\ x\ cosh\ y - 2\ sin\ x\ sinh\ y + 2x +4y$.

$u_{xx} = -\ sin\ x\ cosh\ y - 2\ cos\ x\ sinh\ y +2$.

$u_y = sin\ x\ sinh\ y + 2\ cos\ x\ cosh\ y - 2y + 4x$.

$u_{yy} = sin\ x\ cosh\ y + 2\ cos\ x\ sinh\ y - 2$.

Therefore $u_{xx}+u_{yy} = 0$. Hence u is harmonic.

Now let $\phi_1(x,y) = u_x$ and $\phi_2(x,y) = u_y$.

Therefore $\phi_1(z,0) = cos\ z\ cosh\ 0 - 2\ sin\ z\ sinh\ 0 + 2z$.

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$= \cos z + 2z.$

Similarly $\phi_2(z,0) = 2 \cos z + 4z.$

Therefore $f(z) = \int [\varphi_1(z,0) - i\, \varphi_2(z,0)]\, dz$ (by Milne Thompson Method)

$$= \int[\cos z + 2z - i(2 \cos z + 4z)]dz$$

$$= \sin z + z^2 - 2i \sin z - 2iz^2 + c.$$

Problem 6.3.11

If $f(z) = u(x,y) + iv(x,y)$ is an analytic function and $u(x,y) = \sin 2x / \cosh 2y + \cos 2x$, find $f(z)$.

Solution.

It can be verified that $u(x,y)$ is harmonic.

Now, $u_x = [(\cosh 2y + \cos 2x)\, 2 \cos 2x + 2 \sin^2 2x]/(\cosh 2y + \cos 2x)^2$

$= [2\cosh 2y \cos 2x + 2] / (\cos \quad + \cos 2x)^2$

Also, $u_y = [-2 \sin 2x \sinh 2y] / (\cosh 2y + \cos 2x)^2$

Let $\phi_1(x,y) = u_x$ and $\phi_2(x,y) = u_y$

Therefore $\phi_1(z,0) = [\,2\cosh 0 \cos 2z + 2] / (\cosh 0 + \cos 2z)^2$

$$= 2/\,1+\cos 2z = \sec^2 z.$$

And $\phi_2(z,0) = 0.$

Now, $f(z) = \int [\varphi_1(z,0) - i\, \varphi_2(z,0)]\, dz$

$= \int \sec^2 z\, dz$

$= \tan z + c$

Therefore $f(z) = \tan z + c.$

Problem 6.3.12

Find the analytic function $f(z) = u + iv$ if $u + v = \sin 2x / (\cosh 2y - \cos 2x)$.

Solution.

$u+v = \sin 2x / (\cosh 2y - \cos 2x).$ ....(1)

Therefore,

$u_x + v_x = [\,2(\cosh 2y - \cos 2x)\cos 2x - 2 \sin^2 2x]/(\cosh 2y - \cos 2x)^2$.......(2)

and $u_y + v_y = [\,-2 \sin 2x \sinh 2y]/(\cosh 2y - \cos 2x)^2.$ ....(3)

Space for Hints

Since the required function $f(z) = u+iv$ is to be analytic, u and v satisfy the C-R equations $u_x = v_y$ and $u_y = -v_x$.

Using these equations in (2), we get

$u_x - u_y = [\,2(\cosh 2y - \cos 2x)\cos 2x - 2\sin^2 2x]/(\cos h\, 2y - \cos 2x)^2$.

Therefore,

$u_x(z,0) - u_y(z,0) = [\,2(1 - \cos 2z)\cos 2z - 2\sin^2 2z]/(1 - \cos 2z)^2$.

$=[\,2\cos 2z - 2\,(\sin^2 2z + \cos^2 2z)]/(1 - \cos 2z)^2.$

$= -2(1 - \cos 2z)\,/(1 - \cos 2z)^2.$

$= -2\,/2\sin^2 z = -\operatorname{cosec}^2 z.$ ....(4)

Using C-R equations in (3) ,we get

$u_y + u_x = [\,-2\sin 2x \sinh 2y]/(\cos h\, 2y - \cos 2x)^2.$
Therefore, $u_y(z,0) + u_x(z,0) = 0.$ ....(5)

Now adding (4) and (5) ,we get , $2u_x(z,0) = -\operatorname{cosec}^2 z.$

Therefore $u_x(z,0) = -\frac{1}{2}\operatorname{cosec}^2 z.$ ....(6)

Subtracting (4) from (5) ,we get , $2u_y(z,0) = \operatorname{cosec}^2 z.$

Therefore , $u_y(z,0) = \frac{1}{2}\operatorname{cosec}^2 z.$ ....(7)

Now $f(z) = u(z,0) + iv(z,0)$

$\Rightarrow f'(z) = u_x(z,0) + iv_x(z,0).$

$= u_x(z,0) - iu_y(z,0).$

$= (-1/2)\,(1+i)\operatorname{cosec}2\, z$ [using (6) and (7)]

Integrating with respect to z ,we have,

$f(z) = (\,1+i/2)\cot z + c.$

Problem 6.3.13
Given $v(x,y) = x^4 - 6x^2y^2 + y^4$, find $f(z) = u(x,y) + iv(x,y)$ such that $f(z)$ is analytic.
Solution.

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It can be easily verified that $v(x,y)$ is harmonic.

Now, $v_x = 4x^3 - 12xy^2$ and $v_y = -12x^2y + 4y^3$.

Let $f(z) = u+iv$ be the required analytic function.

By Cauchy-Riemann equations $u_x = v_y$.

Therefore $u_x = -12x^2y + 4y^3$.

Integrating with respect to x, we get, $u = -4x^3y + 4xy^3 + \lambda(y)$ where $\lambda(y)$ is an arbitrary function of y.

Therefore $u_y = -4x^3 + 12xy^2 + \lambda'(y) = -v_x$.

Therefore $-(4x^3 - 12xy^2) = -4x^3 + 12xy^2 + \lambda'(y)$.

Therefore $\lambda'(y) = 0$ so that $\lambda(y) =$ ere c is a constant.

Thus $u = -4x^3y + 4xy^3 + c$.

Therefore $f(z) = (-4x^3 + 4xy^3 + c) + i(x^4 - 6x^2y^2 + y^4)$

$$= i[(x^4 - 6x^2y^2 + y^4) + i(4x^3y - 4xy^3)] + c$$

$$= i(x+iy)^4 + c$$

$$= iz^4 + c.$$

Aliter*(Milne Thompson Method)*

Let $\psi_1(x,y) = v_y$ and $\psi_2(x,y) = v_x$.

Therefore $\psi_1(x,y) = -12x^2y + 4y^3$ and $\psi_2(x,y) = 4x^3 - 12xy^2$.

Therefore $\psi_1(z,0) = 0$ and $\psi_2(z,0) = 4z^3$.

Therefore $f(z) = \int [\psi_1(z,0) + i\,\psi_2(z,0)]\,dz$

$$= i\int 4z^3 dz$$

$$= iz^4 + c.$$

Problem 6.3.14

Find the analytic function $f(z) = u+iv$ given that $u - v = e^x(\cos y - \sin y)$.

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Solution.

$u - v = e^x(\cos y - \sin y)$. ....(1)

Therefore $u_x - v_x = e^x(\cos y - \sin y)$. ....(2)

And $u_y - v_y = - e^x(\sin y + \cos y)$. ....(3)

Since the required function is to be analytic, it has to satisfy the C-R. equations.
Therefore using C-R. equations in (3) , we get

$-v_x - u_x = - e^x(\sin y + \cos y)$. ....(4)

Solving (2) and (4), we get

$u_x = e^x \cos y$ ....(5)

and $v_x = e^x \sin y$ ....(6)

Integrating (6) with respect to x ,we get,

$v = e^x \sin y + f(y)$

Therefore $v_y = e^x \cos y + f'(y)$ ....(7)

Using C-R. equations in (5) and (7), we get , $f'(y) = 0$.

Hence $f(y) = c_1$ where c_1 is a constant.

Therefore $v = e^x \sin y + c_1$.

From (1) $u = e^x \cos y + c_2$.

Now, $f(z) = u+iv$

$= e^x \cos y + ie^x \sin y + c_1 + ic_2$

$= e^x(\cos y + i \sin y) + (c_1 + ic_2)$

$= e^x e^{iy} + \alpha$ (where α is a complex constant)
$= e^{x+iy} + \alpha$
$= e^z + \alpha$.

Problem 6.3.15
If $u+v = (x-y)(x^2 + 4xy + y^2)$ and $f(z) = u+iv$, find the analytic function $f(z)$ in terms of z.

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Solution.

$u+v = (x-y)(x^2 + 4xy + y^2$ ....(1)

Differentiating (1) partially with respect to x, we get,

$u_x+v_x = (x^2 + 4xy + y^2) + (x-y)(2x + 4y)$ ....(2)

Differentiating (1) partially with respect to y, we get,

$u_y+v_y = -(x^2 + 4xy + y^2) + (x-y)(4x + 2y)$ ....(3)

Since $f = u + iv$ is analytic, u and v satisfy the C-R equations.

$u_x = v_y$ *and* $u_y = -v_x$.

Therefore using C-R. equations in (3), we get

$-v_x + u_x = -(x^2 + 4xy + y^2) + (x-y)(4x + 2y)$ ....(4)

Adding (2) and (4), we get, $2u_x = (x-y)(6x + 6y)$.

Therefore $u_x = 3(x^2 - y^2)$ .... (5)

Subtracting (4) from (2), we get, $v_x = 6xy$ ....(6)

Using C-R equations in (6), we get, $u_y = -6xy$ ....(7)

Let $\phi_1(x,y) = u_x$ and $\phi_2(x,y) = u_y$.

Therefore $\phi_1(z,0) = 3z^2$ and $\phi_2(z,0) = 0$.

By Milne-Thompson method,

$f(z) = \int [\varphi_1(z,0) - i\,\varphi_2(z,0)]\,dz$

$= \int 3z^2 dz = z^3 + c.$

Problem 6.3.16

Find the real part of the analytic function whose imaginary part is $e^{-x}[2xy \cos y + (y^2 - x^2) \sin y]$. Construct the analytic function.

Solution.

Let $v = e^{-x}[2xy\cos y + (y^2 - x^2)\sin y]$ and $f(z) = u+iv$ be the required analytic function.

We can prove that v is harmonic. We use Milne-Thompson method to find the harmonic conjugate u of v.

Let $\psi_1(x,y) = v_y = e^{-x}(2x\cos y - 2xy\sin y + 2y\sin y + (y^2 - x^2)\cos y)$

and $\psi_2(x,y) = v_x = e^{-x}(-2xy\cos y - (y^2 - x^2)\sin y + 2y\cos y - 2x\sin y)$.

Therefore $\psi_1(z,0) = e^{-z}(2z - z^2)$ and $\psi_2(z,0) = 0$.

By Milne Thompson method,

$$f(z) = \int [\psi_1(z,0) + i\,\psi_2(z,0)]\,dz$$

$$= \int e^{-z}(2z - z^2)\,dz$$

$$= \int 2ze^{-z}dz - [-z^2e^{-z} + \int e^{-z}\,2z\,dz]$$

$$= z^2e^{-z}$$

$$= (x+iy)^2\,e^{-(x+iy)}$$

$$= [(x^2 - y^2) + 2ixy]\,e^{-x}(\cos y - i\sin y)$$

$$= e^{-x}[(x^2 - y^2) + 2ixy]\,(\cos y - i\sin y)$$

Real part of $f(z) = e^{-x}[(x^2 - y^2)\cos y + 2xy\sin y]$.

(i.e) $u = e^{-x}[(x^2 - y^2)\cos y + 2xy\sin y]$.

Problem 6.3.17

Find the constant a so that $u(x,y) = ax^2 - y^2 + xy$ is harmonic. Find an analytic function $f(z)$ for which u is the real part. Also find its harmonic conjugate.

Solution.

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Space for Hints

$u(x,y) = ax^2 - y^2 + xy$.

Given that u is harmonic. Hence it satisfies Laplace's equation

$$\partial^2u/\partial x^2 + \partial^2u/\partial y^2 = 0.$$

Now, $\partial u/\partial x = 2ax + y$ and $\partial^2u/\partial x^2 = 2a$;

$\partial u/\partial y = -2y + x$ and $\partial^2u/\partial y^2 = -2$.

$\partial^2u/\partial x^2 + \partial^2u/\partial y^2 = 0 \Rightarrow 2a - 2 = 0.$

Hence $a = 1$.

Therefore $u = x^2 - y^2 + xy$.

Hence $u_x = 2x + y$ and $u_y = -2y + x$.

Let $\phi_1(x,y) = u_x = 2x + y$ and $\phi_2(x,y) = u_y = -2y + x$.

Therefore $\phi_1(z,0) = 2z$ and $\phi_2(z,0) = z$.

Therefore $f(z) = \int [\varphi_1(z,0) - i\,\varphi_2(z,0)]\, dz$

$= \int (2z - iz)\, dz$

$= z^2 - iz^2/2 + c$

$= (x+iy)^2 - i(x+iy)^2/2 + c$

$= (x^2 - y^2 + 2ixy) - (i/2)(x^2 - y^2 + 2ixy) + c$

$= (x^2 - y^2 + xy) + i(2xy + (y^2 - x^2)/2) + c.$

Therefore $v(x,y) = 2xy + (y^2 - x^2)/2$ is the harmonic conjugate of $u(x,y)$.

Problem 6.3.18

If $u(x,y)$ is a harmonic function in a region D, prove that

$f(z) = \partial u/\partial x - i\partial u/\partial y$ is analytic in D.

Solution.

Let $U = \partial u/\partial x$ and $V = -\partial u/\partial y$.

Therefore $f(z) = U + iV$. Since u is harmonic, U and V have continuous first order partial derivatives and $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 0$. ....(1)

Also $\partial U/\partial x = \partial^2 u/\partial x^2 = -\partial^2 u/\partial y^2$ [using (1)]

$= \partial V/\partial y$.

Hence $\partial U/\partial x = \partial V/\partial y$.

Now, $\partial U/\partial y = \partial^2 u/\partial y\partial x = \partial^2 u/\partial x\partial y = \partial/\partial x(\partial u/\partial y) = -\partial V/\partial x$.

Hence $\partial U/\partial y = -\partial V/\partial x$.

Thus the partial derivatives of U and V satisfy the Cauchy-Riemann equations. Hence f is analytic in D.

Problem 6.3.1

If u and v are harmonic functions satisfying the Cauchy-Riemann equations in a region D, then $f = u+iv$ is analytic in D.

Solution.

Since u and v are harmonic, the first order partial derivatives of u and v are continuous. Also u and v satisfy the C-R equations in D.

Hence $f = u + iv$ is analytic in D.

Problem 6.3.20

Prove that the real(imaginary) part of an analytic function when expressed in polar form satisfies the equation $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$.

(This equation is the Laplace equation in polar form)

Solution.

We know that Cauchy-Riemann equations in polar form are given by

Space for Hints

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$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad ....(1)$$

and $$\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta} \quad ....(2)$$

We eliminate v from (1) and (2).

Differentiating (1) partially with respect to r and (2) partially with respect to θ, we have

$$\frac{\partial^2 v}{\partial r \partial \theta} = r\frac{\partial^2 u}{\partial r^2} + \frac{\partial u}{\partial r} \quad ....(3)$$

$$\frac{\partial^2 v}{\partial \theta \partial r} = -\frac{1}{r}\frac{\partial^2 u}{\partial \theta^2} \quad ....(4)$$

Since $\frac{\partial^2 v}{\partial r \partial \theta} = \frac{\partial^2 v}{\partial \theta \partial r}$, we have $r\frac{\partial^2 u}{\partial r^2} + \frac{\partial u}{\partial r} = -\frac{1}{r}\frac{\partial^2 u}{\partial \theta^2}$.

$\therefore \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$. Similarly $\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} = 0$.

Problem 6.3.21

ϕ and ψ are functions of x and y satisfying Laplace's equation. If $u = \phi_y - \psi_x$ and $v = \phi_x + \psi_y$, prove that $u + iv$ is analytic.

Solution.

Given that ϕ and ψ satisfy Laplace's equation.

Hence $\phi_{xx} + \phi_{yy} = 0$. ....(1)

And $\psi_{xx} + \psi_{yy} = 0$. ....(2)

Also $u = \phi_y - \psi_x$ and $v = \phi_x + \psi_y$.

Hence $u_x = \phi_{xy} - \psi_{xx}$

$u_y = \phi_{yy} - \psi_{yx}$

$v_x = \phi_{xx} + \psi_{xy}$

$= -\phi_{yy} + \psi_{xy}$ [by(1)]

And $v_v = \phi_{yx} - \psi_{yy}$

$= \phi_{yx} - \psi_{xx}$ [by(2)].

Thus $u_x = v_y$ and $u_y = -v_x$.

Since ϕ and ψ are harmonic, all the partial derivatives are continuous.

Hence $u + iv$ is analytic.

Problem 6.3.22

Show that if u and v are conjugate harmonic functions, the product uv is a harmonic function.

Solution.

Since u and v are conjugate harmonic functions, the product uv is a harmonic functions ,we have

$u_{xx} + u_{yy} = 0.$ ....(1)

$v_{xx} + v_{yy} = 0.$ ....(2)

$u_x = v_y$ ....(3)

$u_y = -v_x$ ....(4)

Now let $\phi = uv$.

$\phi_x = uv_x + vu_x$.

$\phi_{xx} = uv_{xx} + 2u_xv_x + vu_{xx}$

Similarly $\phi_{yy} = uv_{yy} + 2u_yv_y + vu_{yy}$

$= uv_{yy} - 2v_xu_x + vu_{yy}$ [using(3) and (4)].

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Now $\phi_{xx} + \phi_{yy} = u(v_{xx} + v_{yy}) + v(u_{xx}+u_{yy})$

$=0$ [using (1) and (2)]

Therefore $\phi = uv$ is a harmonic function.

Problem 6.3.23

If $f(z)$ is analytic , prove that $(\partial^2/\partial x^2 + \partial^2/\partial y^2)|f(z)|^2 = 4|f'(z)|^2$.

Solution.

Let $f(z) = u+iv$.

Therefore $|f(z)|^2 = u^2 + v^2 = \phi$ (say) and $f'(z) = u_x + iv_x$.

Therefore $\partial^2\phi/\partial x^2 = 2[u_x^2 + uu_{xx} + v_x^2 + vv_{xx}]$ ....(1)

Similarly $\partial^2\phi/\partial y^2 = 2[u_y^2 + uu_{yy} + v_y^2 + vv_{yy}]$ ....(2)

Since u and v are harmonic,

$u_{xx} + u_{yy} = 0$ *and* $v_{xx} + v_{yy} = 0$ ....(3)

Adding (1) and (2) using (3) we get

$$\partial^2\phi/\partial x^2 + \partial^2\phi/\partial y^2 = 4(u_x^2 + v_x^2)$$

$$= 4|u_x + iv_x|^2$$

$$= 4|f'(z)|^2.$$

Hence the result.

Problem 6.3.24

If $f(z) = u+iv$ is analytic and $f(z) \neq 0$, prove that

(i) $(\partial^2/\partial x^2 + \partial^2/\partial y^2)\log|f(z)| = 0.$

(ii) $\nabla^2 \operatorname{amp} f(z) = 0.$

Solution.

$\log f(z) = \log|f(z)| + i \operatorname{amp} f(z).$

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Since $f(z) \neq 0$, $\log|f(z)|$ exists.

Further since $f(z)$ is analytic and $f(z) \neq 0$, $\log f(z)$ is also analytic.

Therefore $\log|f(z)|$ and amp $f(z)$ are the real and imaginary parts of the analytic function $\log f(z)$.

Hence both $\log|f(z)|$ and $amp\, f(z)$ satisfy Laplace equation.

$$(i)\ \partial^2/\partial x^2 (\log|f(z)|) + \partial^2/\partial y^2 (\log|f(z)|) = 0$$

$$(i.e)\quad (\partial^2/\partial x^2 + \partial^2/\partial y^2)(\log|f(z)|) = 0.$$

(ii) Also, $\partial^2/\partial x^2 (amp\, f(z)) + \partial^2/\partial y^2 (amp\, f(z)) = 0$

$$(i.e)\quad (\partial^2/\partial x^2 + \partial^2/\partial y^2)(amp\, f(z)) = 0.$$

$$(i.e)\quad \nabla^2\ amp\, f(z) = 0.$$

Problem 6.3.25

Given the function $w = z^3$ where $w = u+iv$. Show that u and v satisfy the Cauchy-Riemann equations. Prove that the families of curves $u = c_1$ and $v = c_2$ (c_1 and c_2 are constants) are orthogonal to each other.

Solution.

$$w = z^3 = (x+iy)^3$$

$$= (x^3 - 3xy^2) + i\,(3x^2y - y^3).$$

Therefore $u = (x^3 - 3xy^2)$ and $v = (3x^2y - y^3)$.

$u_x = 3x^2 - 3y^2$ and $u_y = -6xy$;

$v_x = 6xy$ and $v_y = 3x^2 - 3y^2$.

We note that $u_x = v_y$ and $u_y = -v_x$.

Hence u and v satisfy the Cauchy-Riemann equations.

Now $u_{xx} = 6x$ and $u_{yy} = -6x$.

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Therefore $u_{xx} + u_{yy} = 6x - 6x = 0.$

Hence u satisfies the Laplace equations.

Similarly $v_{xx} + v_{yy} = 6y - 6y = 0.$

Hence v satisfies the Laplace equations.

$$u = c_1 \Rightarrow x^3 - 3xy^2 = c_1$$

Differentiating with respect to x, we get ,

$3x^2 - 3(2xy\,(dy/dx) + y^2) = 0.$

Therefore $dy/dx = 3(x^2 - y^2)/6xy = (x^2 - y^2)/2xy.$

Therefore slope of the tangent at (x_0, y_0) for the curve $u = c_1$ is given by

$m_1 = x_0^2 - y_0^2/2x_0y_0.$

Now $v = c_2 \Rightarrow 3x^2y - y^3 = c_2.$

Differentiating with respect to x, we get

$$3(2xy + x^2(dy/dx)) - 3y^2\,(dy/dx) = 0.$$

Hence $dy/dx(3x^2 - 3y^2) = -6xy.$

Therefore $dy/dx = -2xy/(x^2 - y^2).$

Slope of the tangent to the curve $u = c_2$ at (x_0, y_0) is given by

$m_2 = -2x_0y_0/(x_0^2 - y_0^2).$

Clearly, $m_1m_2 = -1.$

Therefore the two families of curves are orthogonal.

Exercises 6.3.26

1. Prove that the following functions re harmonic. Also find a harmonic conjugate.

   (i) $u = \sinh x \sin y.$

(ii) $u = 3x^2y + 2x^2 - y^3 - 2y^2$.

(iii) $u = e^x \cos y$.

2. Prove that the following functions are harmonic. Also find a function v such that $f(z) = u+iv$ is analytic and express $f(z)$ in terms of z.

(i) $u = 2x(1-y)$

(ii) $u = e^x(x \cos y - y \sin y)$

(iii) $u = 2xy + 3y$.

(iv) $u = y/(x^2+y^2)$.

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UNIT 7

7.0 CONFORMAL MAPPING AND SOME DEFINITIONS

In this section, we study the concept of bilinear transformations, cross ratio and fixed points. We start with the necessary definitions which are useful throughout this material.

Definition 7.0.1

A ***curve*** C in the complex plane is given by a continuous function

$\gamma: [a,b] \rightarrow \mathbf{C}$.

If $\gamma(t) = x(t) + iy(t)$, then the curve C is determined by the two continuous real valued functions of the real parameter t given by $x = x(t)$ and $y = y(t)$ where $a \leq t \leq b$. We also write $z = z(t) = x(t) + iy(t)$ where $a \leq t \leq b$. The point $z(a)$ is called the ***origin*** of the curve and $z(b)$ is called the ***terminus*** of the curve.

The curve C is said to be ***simple*** if $t_1 \neq t_2 \Rightarrow z(t_1) \neq z(t_2)$.

Equivalently C is simple if the function γ is one-one.

The curve C is called a ***closed curve*** if $z(a) = z(b)$ and C is called a ***simple closed curve*** if (i) $z(a) = z(b)$ (ii) $z(t_1) \neq z(t_2)$ for any other pair of distinct real numbers $t_1, t_2 \in [a,b]$.

A simple closed curve is also called a ***Jordan curve***.

A curve C is said to be ***differentiable*** if $z'(t)$ exists and is continuous. If further $z'(t) \neq 0$, then the curve is said to be ***regular (smooth)***.

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Geometrically the regular curve has a tangent whose direction is determined by the argument of $z'(t)$.

If C is a curve determined by the equation $z = z(t)$ where $a \leq t \leq b$, then the opposite cure of C denoted by $-C$ us given by the equation $z(t) = z(b+a-t)$ where $a \leq t \leq b$.

Example 7.0.2

The polygonal line given by

$$z(t) = \begin{cases} t + it & \text{if } 0 \leq t \leq 1 \\ t + i & \text{if } 1 \leq t \leq 2 \end{cases}$$

consisting of a line segment from 0 to $1+i$ followed by another line segment from $1+i$ to $2+i$ is a simple curve..

We notice that the above curve is differentiable except a $1+i$. Such a curve is called a ***piecewise differentiable curve***.

Definition 7.0.3

A curve C given by $z = z(t)$ is said to be ***piecewise differentiable*** if it is differentiable except at a finite number of points and at any point where $z(t)$ is not differentiable, it has a left derivative and right derivative.

Example 7.0.4

The equation given by $z(t) = \cos t + i \sin t$, where $0 \leq t \leq 2\pi$ represents the unit circle C with centre O and radius 1 described in the anticlockwise

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direction. The origin and terminus of the curve are $z(0) = 1 = z(2\pi)$.

The same circle with negative orientation $-C$ is given by the equation

$z(t) = \cos(2\pi - t) + i \sin(2\pi - t)$. This is a simple closed curve.

Example 7.0.5

In general the equation $z(t) = a + r(\cos t + i \sin t)$ where $0 \leq t \leq 2\pi$ represents a positively oriented circle with centre a and radius r. This is also a simple closed curve.

Example 7.0.6

The curve represented by $z(t) = \cos t + i \sin t$ where $0 \leq t \leq 4\pi$ is a closed curve. However it is not a simple closed curve, since $z(\pi/2) = z(5\pi/2)$. Actually the equation represents a unit circle traversed twice.

Definition 7.0.7

Let f be an analytic function in a region D. Let C be a curve given by the equation $z = z(t)$ where $a \leq t \leq b$ and lying in D.

Then the equation $w = w(t) = f(z(t))$ defines another curve C' in the w-plane and is called the ***image of the curve C*** under f.

Definition 7.0.8

Let f be a continuous function defined in the region D. Let $z_0 \in D$. Let C_1 and C_2 be two regular curves passing through z_0 and lying in D. Let C'_1 and C'_2 be the images of C_1 and C_2 respectively under f. If the angle between C_1 and C_2 is equal to the angle between C_1' and C_2' both in magnitude and direction, then f is said to be ***conformal*** at z_0.

Thus a conformal mapping preserves angle both in magnitude and direction.

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If the angle is preserved only in magnitude and direction is reversed, then the mapping is said to be ***isogonal*** or ***indirectly conformal.***

Definition 7.0.9

Let $f(z)$ be an analytic function defined in D and let $z_0 \in D$. Then z_0 is called a ***critical point*** of $f(z)$ if $f'(z_0) = 0$.

BILINEAR TRANSFORMATIONS

INTRODUCTION

A function $f: \mathbf{C} \rightarrow \mathbf{C}$ can be thought of as a transformation from one complex plane to another complex plane. Hence the nature of a complex function can be described by the manner in which it maps regions and curves from one complex plane to another. In this unit, we shall discuss bilinear transformations and see how various regions are transformed by these transformations.

7.1 ELEMENTARY TRANSFORMATIONS

1. **Translation:** $w = z + b$.

Consider the transformation $w = z + b$.

If $z = x+iy$, $w = u+iv$ and $b = b_1+ib_2$, then the image of the point (x,y) in the z-plane is the point $(x+b_1, y+b_2)$ in the w-plane.

Under this transformation, the image of any region is simply a translation of that region. Hence the two regions have the same

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shape, size and orientation. In particular the image of a straight line is a straight line and the image of a circle with centre a and radius r is a circle with centre $a + b$ and radius r.

We note that ∞ is the only fixed point of this transformation when $b \neq 0$.

2. **Rotation:** $w = az$ where $|a| = 1$.

Consider the transformation $w = az$ where $|a| = 1$.

Let $z = re^{i\theta}$ and $a = e^{i\alpha}$ so that $|a| = 1$.

Therefore, $w = az = e^{i\alpha}(re^{i\theta}) = re^{i(\theta+\alpha)}$.

Therefore, a point with polar coordinates (r,θ) in the z-plane is mapped to the point $(r, \theta+\alpha)$ in the w-plane. Hence this transformation represents a ***rotation*** through an angle $\alpha = \arg a$ about the origin.

Under this transformation also straight lines are mapped into straight lines and circles are mapped into circles.

We note that 0 and ∞ are the two fixed points of this transformation.

3. **Magnification or Contraction:** $w = bz$ $(b>0$,real)

Consider the transformation $w = bz$ where b is real and $b>0$.

Then a point with polar coordinates (r,θ) in the z-plane is mapped into the point (br,θ) in the w-plane. Hence this transformation represents a ***magnification*** or ***contraction*** by the factor according as $b>1$ or $b<1$.

Under this transformation also straight lines are mapped into straight lines and circles are mapped into circles.

We note that 0 and ∞ are the fixed points of this transformation.

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Note that in general the transformation $w = bz$ where b is a non-zero complex number represents a rotation through an angle arg b followed by a magnification or a contraction by the factor $|b|$. Such a transformation is called a ***homothetic transformation.***

4. **Inversion:** $w = 1/z$

Consider the transformation $w = 1/z$

Put $z = re^{i\theta}$.

Therefore $w = (1/r)e^{-i\theta}$.

This transformation can be expressed as a product of two transformations

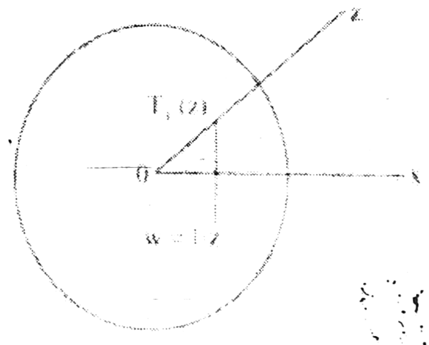
$T_1(z) = (1/r)e^{i\theta}$ and $T_2(z) = re^{-i\theta} = \bar{z}$.

For, $(T_1 \circ T_2)(z) = T_1(T_2(z))$

$= T_1(re^{-i\theta})$

$= (1/r)e^{-i\theta} = 1/z.$

The transformation $T_1(z) = (1/r)e^{i\theta}$ represents the ***inversion*** with respect to the unit circle $|z| = 1$ and $T_2(z) = \bar{z}$ represents ***reflection*** about the real axis.



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Hence the transformation $w = 1/z$ is the inversion with respect to the unit circle followed by the reflection about the real axis.

Here points outside the unit circle are mapped into points inside the unit circle and vice versa. Points on the circle are reflected about the real axis.

In terms of Cartesian coordinates the above transformation can be expressed in the form

$$w = u+iv = 1/(x+iy) = (x - iy)/(x^2 + y^2).$$

Therefore $u = x/(x^2 + y^2)$ and $v = -y/(x^2 + y^2)$.

Similarly from $z = 1/w$ we get $x = u/(u^2 + v^2)$ and $y = -v/(u^2 + v^2)$ ...(1)

Now, consider the equation $a(x^2 + y^2) + bx + cy + d = 0$

where a, b, c, d are real. ....(2)

This equation represents a circle or a straight line according as $a \neq 0$ or $a = 0$. Using (1) and (2), we get,

$d(u^2 + v^2) + bu - cv + a = 0.$ ....(3)

Now, suppose $a \neq 0$; $d \neq 0$.

In this case, both (2) and (3) represent circles not passing through the origin. Hence circles not passing through the origin are mapped into circles not passing through the origin.

Similarly, a circle passing through the origin is maped into a straight line not passing through the origin.

A straight line not passing through the origin is mapped into a circle passing through the origin.

A straight line passing through the origin is again mapped into a line passing through the origin.

Thus we see that **under the transformation w = 1/z the image of a circle need not be a circle and the image of a straight line need not be a straight line.**

However the family of circles and lines are again mapped into the family of circles and lines.

We note that the fixed points of the transformation w = 1/z are 1 and -1.

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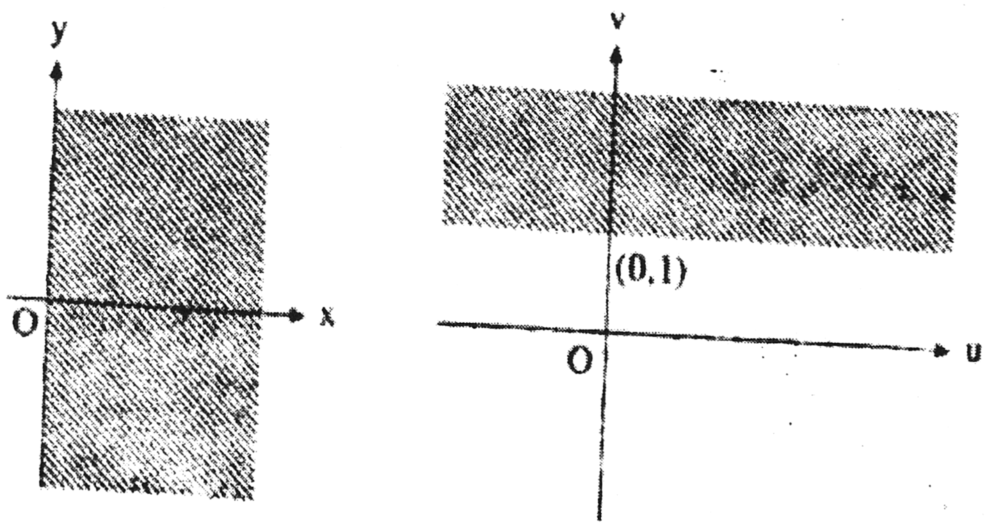
Solved Problems

Problem 7.1.1

Under the transformation $w = iz + i$, show that the half plane $x > 0$ maps onto the half plane $v > 1$.

Solution.

Let $z = x+iy$ and $w = u+iv$



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$w = iz + i \Rightarrow w = i(x+iy) + i = -y + i(x+1)$.

Therefore $u+iv = -y + i(x+1)$.

Therefore $u = -y$ and $v = x+1$.

Therefore $x > 0 \Leftrightarrow v > 1$.

Therefore the half plane $x > 0$ is mapped into the half plane $v > 1$.

Problem 7.1.2

Show that the region in the z-plane given by $x > 0$ and $0 < y < 2$ is mapped into the region in the w-plane given by $-1 < u < 1$ and $v > 0$ under the transformation $w = iz + 1$.

Solution.

Let $z = x+iy$ and $w = u+iv$.

$w = iz + 1 \Rightarrow w = i(x+iy) + 1$

$\Rightarrow u+iv = (-y + 1) + ix$.

Therefore $u = 1 - y$ and $v = x$.

Therefore $x > 0$ and $0 < y < 2 \Leftrightarrow v > 0$ and $-1 < u < 1$. Hence the given region is mapped in to the region v>0 and $-1 < u < 1$.

Problem 7.1.3

Find the image of the square region with vertices $(0,0)$, $(2,0)$, $(2,2)$, $(0,2)$ under the transformation $w = (1+i)z + (2+i)$.

Solution.

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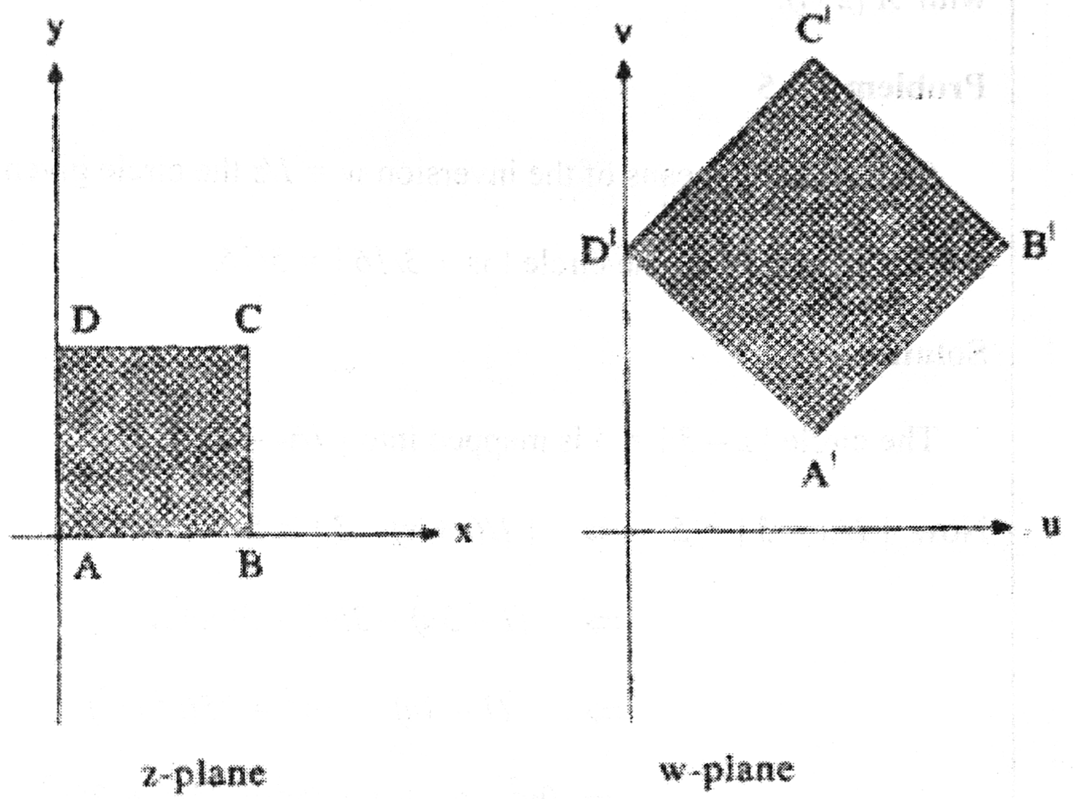

$w = (1+i)z + (2+i)$. Under the transformation

A$(0,0)$ is mapped into A$' = (1+i)(0+0i) + (2+i) = 2+i$.

B$(2,0)$ is mapped into $B' = (1+i)(2+0i) + (2+i) = 4+3i$.

C$(2,2)$ is mapped into $C' = (1+i)(2+2i) + (2+i) = 2+5i$.

D$(0,2)$ is mapped into $D' = (1+i)(0+2i) + (2+i) = 0+3i$.

Therefore the required image region is another square $A'B'C'D'$ as given in the figure.

Note 7.1.4

Since the given transformation is combination of $w_1 = (1+i)z$ and the translation $w = w_1 + (2+i)$, we get the image of the square $ABCD$ by increasing each length of $ABCD$ by $\sqrt{2}$ times and rotating the square

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through an angle 45° about A and then translating it such that A coincides with $A'(2+i)$.

Problem 7.1.5

Show that by means of the inversion $w = 1/z$ the circle given by $|z-3| = 5$ is mapped into the circle $|w + 3/16| = 5/16$.

Solution.

The circle $|z-3| = 5$ is mapped into $|1/w - 3| = 5$.

Now $|1/w-3| = 5 \Rightarrow |1/(u+iv) - 3| = 5$

$$\Rightarrow |(1-3u) - 3iv| = 5|u+iv|$$

$$\Rightarrow (1-3u)^2 + 9v^2 = 25(u^2+v^2)$$

$$\Rightarrow 9u^2 - 6u + 1 + 9v^2 = 25u^2 + 25v^2$$

$$\Rightarrow 16(u^2+v^2) + 6u - 1 = 0.$$

$$\Rightarrow u^2 + v^2 + (6/16)\,u - 1/16 = 0.$$

This is a circle with centre $(-3/16,0)$ and radius $\sqrt{\left(\frac{3}{16}\right)^2 + \frac{1}{16}} = \frac{5}{16}$.

Hence the image circle in the w-plane is given by the equation

$$\left|w + \frac{3}{16}\right| = \frac{5}{16}.$$

Problem 7.1.6

Find the image of the circle $|z - 3i| = 3$ under the map $w = 1/z$.

Solution.

The image of the circle $|z-3i| = 3$ under the transformation $w = 1/z$ is given by the equation $|1/w - 3i| = 3$.

Now, $|1/w - 3i| = 3 \Rightarrow |1/(u+iv) - 3i| = 3$

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$$\Rightarrow \quad |1 - 3i(u+iv)| = 3|u+iv|$$

$$\Rightarrow \quad |(1+3v) - i3u| = 3|u+iv|$$

$$\Rightarrow \quad (1+3v)^2 + 9u^2 = 9(u^2+v^2)$$

$$\Rightarrow \quad 1 + 6v + 9v^2 + 9u^2 = 9u^2 + 9v^2$$

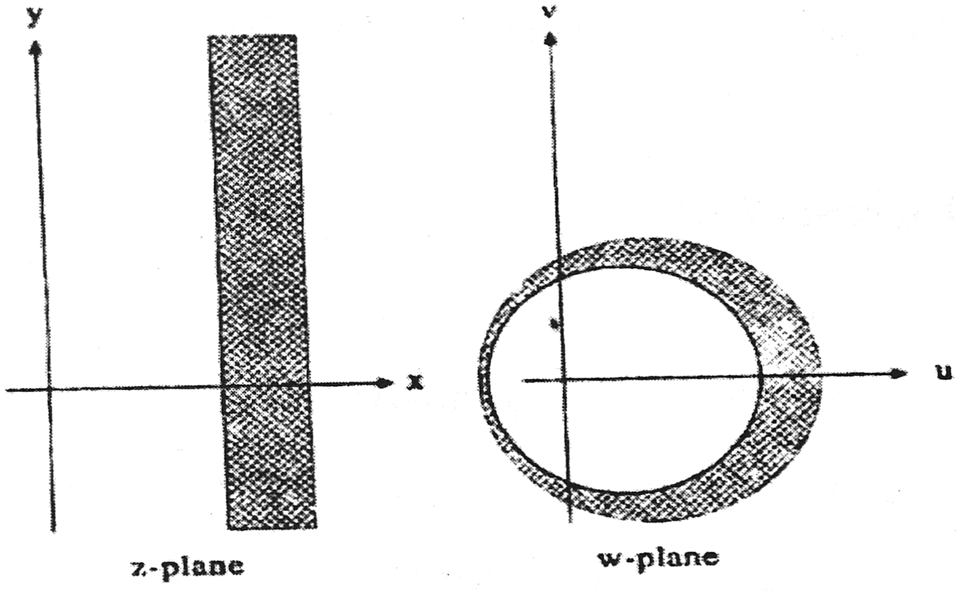
$$\Rightarrow \quad 6v + 1 = 0.$$

Hence the image of the circle $|z - 3i| = 3$ under $w = 1/z$ in the z-plane is the straight line $6v + 1 = 0$ in the w-plane.

Problem 7.1.7

Find the image of the strip $2 < x < 3$ under $w = 1/z$.

Solution.



The transformation $w = 1/z$ can be written in Cartesian coordinates as

$$x = \frac{u}{u^2+v^2} \quad and \quad y = \frac{-v}{u^2+v^2}.$$

Now, $x > 2 \Rightarrow \dfrac{u}{u^2+v^2} > 2.$

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$$\Rightarrow\ 2(u^2+v^2) - u < 0.$$

$$\Rightarrow\ u^2+v^2-u/2\ < 0.$$

$\therefore$ The region $x > 2$ is mapped into a region represented by $u^2+v^2-u/2 < 0$, which is the interior of the circle with centre $\left(\frac{1}{4},0\right)$ and radius ½.

$$\text{Now, } x < 3 \Rightarrow \frac{u}{u^2+v^2} < 3$$

$$\Rightarrow 3(u^2+v^2) - u > 0$$

$$\Rightarrow\ u^2+v^2-u/3\ > 0.$$

$\therefore$ The region $x < 3$ is mapped onto the exterior of the circle with centre $(1/6,0)$ and radius $\frac{1}{\sqrt{6}}$

$\therefore$ The strip $2 < x < 3$ is mapped onto the region bounded by the circle $u^2+v^2 = u/2$ *and* $u^2+v^2 = u/3$ in the w-plane.

Exercises 7.1.8

1. Find the image of the square region with vertices $(0,0)$, $(1,0)$ and $(0,1)$ under the transformation $w = z + (3-i)$.
2. Find the image of the rectangular region bounded by $x = 0; y = 0; x = 2;\quad y = 1$under the (i) translation $w = z + (1-2i)$ (ii) rotation $w = iz$ (iii) transformation $w = (1+i)z + (2-i)$.
3. Find the image of the strip $0 < x < 1$ under the transformation $w = iz$.
4. Find the image of the region $y > 1$ under the transformation $w = iz + 1$.

Answers: 2.(i)The rectangular region bounded by $u=1, v=-2, u=3$ and $v=-1$

(ii) The rectangular region bounded by $u=0, v=0, u=-6$ and $v=2$

(iii) The rectangular region bounded by $u+v=1, v-u=-3, u+v=5$ and $v-u=-1$

3. The strip $0<v<1$.

4. $u+v>2$.

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7.2 BILINEAR TRANSFORMATIONS

Definition 7.2.1

A transformation of the form $w = T(z) = \dfrac{az+b}{cz+d}$ ....(1)

Where a,b,c,d are complex constants and $ad - bc \neq 0$, is called a ***bilinear transformation*** or ***Mobius transformation.***

We define $T(\infty) = \dfrac{a}{c}$ and $T(\dfrac{-d}{c}) = \infty$. Hence T becomes a 1 – 1 onto map of the extended complex plane onto itself.

The inverse of (1) is given by

$$z = T^{-1}(w) = \frac{-dw+b}{cw-a}$$

which is also a bilinear transformation.

Theorem 7.2.2

Any bilinear transformation can be expressed as a product of translation, rotation, magnification or contraction and inversion.

Proof.

$Let\ w = \dfrac{az+b}{cz+d}$ where $ad - bc \neq 0$ ....(1)

be the given bilinear transformation.

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Case i. $c = 0$.

Hence $d \neq 0$ (since $ad - bc \neq 0$),

$$\therefore \quad (1) \Rightarrow w = \frac{az+b}{d}$$

$$= (a/d)z + (b/d).$$

Now, let $T_1(z) = (a/d)z$ and $T_2(z) = z + (b/d)$.

T_1 and T_2 are elementary transformations and

$$(T_2 \circ T_1)(z) = T_2[(a/d)z] = (a/d)z + (b/d)$$

$$= T(z).$$

Case ii. $c \neq 0$.

$$w = \frac{az+b}{cz+d}$$

$$= \frac{a[z+(d/c)]+b-(ad/c)}{c[z+(d/c)]}$$

$$= \frac{a}{c} + \frac{b-(ad/c)}{cz+d}.$$

Now, let $T_1(z) = cz + d$

$$T_2(z) = 1/z.$$

$$T_3(z) = \left(b - \frac{ad}{c}\right)z.$$

$$T_4(z) = z + (a/c)$$

Then $T(z) = (T_4 \circ T_3 \circ T_2 \circ T_1)(z)$.

Hence the theorem.

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Corollary 7.2.3

Under a bilinear transformation, the family of circles and lines are transformed into the family of circles and lines.

Proof.

Under each of the elementary transformations, the family of circles and straight lines are transformed into the family of circles and lines. Hence the result follows.

Solved Problems

Problem 7.2.4

Show that the transformation $w = \frac{5-4z}{4z-2}$ maps the unit circle $|z| = 1$ into a circle of radius unity and centre $\frac{-1}{2}$.

Solution.

$$w = \frac{5-4z}{4z-2}$$

$\therefore 4wz - 2w = 5 - 4z.$

$\therefore (4w + 4)z = 5 + 2w.$

$\therefore z = \frac{5+2w}{4w+4}.$

Now, $|z| = 1 \Rightarrow z\bar{z} = 1.$

$$\Rightarrow \left(\frac{5+2w}{4w+4}\right)\left(\frac{5+2\bar{w}}{\overline{4w+4}}\right) = 1.$$

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$$\Rightarrow 25+4w\bar{w}+10w+10\bar{w}=16w\bar{w}+16+16(w+\bar{w})$$

$$\Rightarrow 12w\bar{w}+6\bar{w}+6w-9=0$$

$$\Rightarrow w\bar{w}+\frac{1}{2}\bar{w}+\frac{1}{2}w-\frac{3}{4}=0.$$

This represents the equation of the circle with centre $\frac{-1}{2}$ and radius $\sqrt{\frac{1}{4}+\frac{3}{4}}=1.$

Hence the result.

Problem 7.2.5

Show that the transformation w = $\frac{2z+3}{z-4}$ maps the circle $z\bar{z}-2(z+\bar{z})=$ 0 into a straight line given by $2(w+\bar{w})+3=0.$

Solution.

$$w=\frac{2z+3}{z-4}.$$

$\therefore$ $w(z-4)=2z+3.$

$\therefore$ $z(w-2)=3+4w.$

$$\therefore z=\frac{3+4w}{w-2}$$

The image of the circle $z\bar{z}-2(z+\bar{z})=0$ is

$$\left(\frac{3+4w}{w-2}\right)\left(\frac{3+4\bar{w}}{\bar{w}-2}\right)-2\left[\frac{3+4w}{w-2}+\frac{3+4\bar{w}}{\bar{w}-2}\right]=0.$$

On simplification, we get $2(w+\bar{w}) + 3 = 0$ which is obviously a straight line.

Problem 7.2.6

Show that $w = \frac{z-1}{z+1}$ maps the imaginary axis in the z-plane onto the circle $|w| = 1$. What portion of the z-plane corresponds to the interior of the circle $|w| = 1$.

Solution.

$$|w| = 1 \Leftrightarrow \left|\frac{z-1}{z+1}\right| \text{ ,since } w = \frac{z-1}{z+1}$$

$$\Leftrightarrow |z-1| = |z+1|$$

$$\Leftrightarrow |x+iy-1| = |x+iy+1|$$

$$\Leftrightarrow (x-1)^2 + y^2 = (x+1)^2 + y^2$$

$$\Leftrightarrow x = 0.$$

Hence the transformation $w = \frac{z-1}{z+1}$ maps the imaginary axis x = 0 onto the unit circle $|w| = 1$.

Also since the point $z = 1$ is mapped to $w = 0$,it follows that the half plane $x > 0$ is mapped onto the interior of the circle $|w| = 1$.

Exercises 7.2.7

1. Express each of the following bilinear transformations as a product of elementary transformations.

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(i) $w = \dfrac{z-1}{z+1}$

(ii) $w = \dfrac{3z+2i}{iz+6}$

(iii) $w = \dfrac{1+i-3z}{2-i+iz}$

2. Prove that the set of all bilinear transformations forms a group under composition of mappings.

3. Show that the transformation $w = \dfrac{i-iz}{1+z}$ maps the unit circle $|z| = 1$ into the real axis of the plane.

7.3 CROSS RATIO

Definition 7.3.1

Let z_1, z_2, z_3, z_4 be four distinct points in the extended complex plane.

The cross ratio of these four points denoted by (z_1, z_2, z_3, z_4) is denoted by

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$$(z_1,z_2,z_3,z_4) = \begin{cases} \dfrac{(z_1-z_3)(z_2-z_4)}{(z_1-z_4)(z_2-z_3)} & \text{if none of } z_1,z_2,z_3,z_4 \text{ is } \infty \\ \dfrac{z_1-z_3}{z_1-z_4} & \text{if } z_2 \text{ is } \infty \\ \dfrac{z_2-z_4}{z_1-z_4} & \text{if } z_3 \text{ is } \infty \\ \dfrac{z_1-z_3}{z_2-z_3} & \text{if } z_4 \text{ is } \infty \\ \dfrac{z_2-z_4}{z_2-z_3} & \text{if } z_1 \text{ is } \infty \end{cases}$$

Theorem 7.3.2

Any bilinear transformation preserves cross ratio.

Proof.

Let $w = \dfrac{az+b}{cz+d}$, $ad-bc \neq 0$ be the given bilinear transformation. Let z_1, z_2, z_3, z_4 be four distinct points. Let their images under this transformation be w_1, w_2, w_3, w_4 respectively.

We assume that all the z_i and w_i are different from ∞.

We claim that $(z_1, z_2, z_3, z_4) = (w_1, w_2, w_3, w_4)$

We have $w_i = \dfrac{az_i+b}{cz_i+d}$ $(i = 1,2,3,4)$.

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Now, $w_1 - w_3 = \dfrac{az_1 + b}{cz_1 + d} - \dfrac{az_3 + b}{cz_3 + d}$

$$= \frac{(ad - bc)(z_1 - z_3)}{(cz_1 + d)(cz_3 + d)} = k_1(z_1 - z_3) \quad (say)$$

Similarly, $w_2 - w_4 = k_2(z_2 - z_4)$.

$\therefore (w_1 - w_3)(w_2 - w_4) = k_1k_2(z_1 - z_3)(z_2 - z_4)$.

$$= k(z_1 - z_3)(z_2 - z_4).$$

Similarly, we can prove that

$$(w_1 - w_4)(w_2 - w_3) = k(z_1 - z_4)(z_2 - z_3).$$

$$\therefore \frac{(w_1 - w_3)(w_2 - w_4)}{(w_1 - w_4)(w_2 - w_3)} = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)}$$

The proof is similar if one of the z_i or w_i is ∞.

Note 7.3.3

Four distinct points z_1, z_2, z_3, z_4 are collinear or concyclic iff (z_1, z_2, z_3, z_4) is real.

Further any bilinear transformation preserves cross ratio. Hence it follows that the family of circles and straight lines are mapped into the family of circles and straight lines.

Note 7.3.4

The bilinear transformation which maps the three points z_1, z_2, z_3 to three points w_1, w_2, w_3 is given by

$$(z, z_1, z_2, z_3) = (w, w_1, w_2, w_3).$$

Solved Problems

Problem 7.3.5

Find the bilinear transformation which maps the points $z_1 = 2, z_2 = i, z_3 = -2$, onto $w_1 = 1, w_2 = i, w_3 = -1$ respectively.

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Solution.

Let the image of any point z under the required transformation be w.

The required bilinear transformation is given by equation

$$(w, 1, i, -1) = (z, 2, i, -2)$$

$$\therefore \quad \frac{(w-i)(1+1)}{(w+1)(1-i)} = \frac{(z-i)(2+2)}{(z+2)(2-i)}$$

$$\therefore \frac{2(w-i)}{(w+1)(1-i)} = \frac{4(z-i)}{(z+2)(2-i)}$$

$$\therefore \frac{(w-i)}{w-iw+1-i} = \frac{2(z-i)}{2z-iz+4-2i}$$

$$\therefore iwz + 6w - 3z - 2i = 0.$$

$$\therefore w(iz + 6) = 3z + 2i.$$

$$\therefore w = \frac{3z+2i}{iz+6}.$$

This is the required bilinear transformation.

Problem 7.3.6

Find the bilinear transformation which maps the points $z = -1, 1, \infty$ respectively on $w = -i, -1, i$.

Solution.

Let the image of any point z under the required bilinear transformation be w. Since bilinear transformation preserves cross ratio, we have,

$$(z, -1, 1, \infty) = (w, -i, -1, i,).$$

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$$\therefore \frac{(z-1)}{-1-1} = \frac{(w+1)(-i-i)}{(w-i)(-i+1)}.$$

$\Rightarrow (z-1)(w-iw-i-1) = 4iw+4$

$\Rightarrow w(z-1-i(z-1)-4i) = 4i+(i+1)(z-1)$

$\Rightarrow w = \dfrac{(i+1)z+3i-1}{(1-i)z-3i-1}$ which is the required bilinear transformation.

Problem 7.3.7

Determine the bilinear transformation which maps $0, 1, \infty$ into $i, -1, -i$ respectively. Under this transformation show that the interior of the unit circle of the z –plane maps onto the half plane left to the v axis (left halt of the w-plane).

Solution.

The required bilinear transformation is given by the equation

$$(w, i, -1, -i) = (z, 0, 1, \infty).$$

$$\therefore \frac{(w+1)(i+i)}{(w+i)(i+1)} = \frac{(z-1)}{(0-1)}.$$

$$\therefore \frac{2i(w+1)}{iw+w-1+i} = 1-z.$$

$$\therefore 2iw + 2i = iw + w - 1 + i - iwz - wz + z - iz.$$

$$\therefore w(2i - i - 1 + iz + z) = z - iz - 2i - 1 + i.$$

$$\therefore w[(i-1) + (i+1)z] = z(1-i) - (1+i).$$

$$\therefore w = \frac{z(1-i)-(1+i)}{z(1+i)-(1-i)}$$

$$= \frac{z - \left(\frac{1+i}{1-i}\right)}{z - \left(\frac{1-i}{1+i}\right)}$$

$$= \frac{z-i}{z-(1/i)}$$

$$= \frac{z-i}{z+i}.$$

$\therefore$ The required bilinear transformation is $w = \frac{z-i}{z+i}$.

The equations of the left half of the w-plane and the interior of the unit circle in the z-plane are $Re\, w < 0$ and $|z| < 1$ respectively. Now,

$$Re\, w < 0 \quad \Leftrightarrow \quad Re\left(\frac{z-i}{z+i}\right) < 0$$

$$\Leftrightarrow \quad Re\left[\frac{(z-i)(\bar{z}+\bar{i})}{|z+i|^2}\right] < 0$$

$$\Leftrightarrow \quad Re\,(z-i)(\bar{z}-i) < 0$$

$$\Leftrightarrow \quad Re\,(z\bar{z} - i(z+\bar{z}) - 1) < 0$$

$$\Leftrightarrow \quad Re\,(z\bar{z}) - 1 < 0 \quad (\because\ i(z+\bar{z}) \text{ is imaginary})$$

$$\Leftrightarrow \quad |z|^2 < 1$$

$$\Leftrightarrow \quad |z| < 1.$$

$\therefore$ The left half plane is mapped into the interior of the unit circle.

Problem 7.3.8

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Find the bilinear transformation which maps $-1, 0, 1$ of the z-plane onto $-1, -i, 1$ of the w-plane. Show that under this transformation, the upper half of the z-plane maps onto the interior of the unit circle $|w| = 1$.

Solution.

The required bilinear transformation is given by the equation

$(w, -1, -i, 1) = (z, -1, 0, 1)$.

$$\therefore \frac{(w+i)(-1-1)}{(w-1)(-1+i)} = \frac{(z-0)(-1-1)}{(z-1)(-1-0)}.$$

$$(i.e)\ \frac{-2(w+i)}{(w-1)(i-1)} = \frac{-2z}{(1-z)}.$$

$\therefore (1-z)(w+i) = z(w-1)(i-1)$.

$(i.e)\ w + i = iwz + z$.

$(i.e)\ w(1-iz) = z - i$.

$$\therefore w = \frac{z-i}{1-iz} = \frac{i(z-i)}{i(1-iz)} = i\left(\frac{z-i}{z+i}\right).$$

$\therefore$ The required bilinear transformation is $w = i\left(\dfrac{z-i}{z+i}\right)$.

The equations of the upper half of the w-plane and the interior of the unit circle in the z-plane are given by $Im\ w > 0$ and $|z| < 1$ respectively.

$$\text{Now, Im } w > 0 \quad \Leftrightarrow \quad Im\left(i\left(\frac{z-i}{z+i}\right)\right) > 0$$

$$\Leftrightarrow \quad Re\left(\frac{z-i}{z+i}\right) < 0$$

$$\Leftrightarrow \quad |z| < 1.$$

Hence the upper half plane is mapped into the interior of the unit circle.

Exercises.

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1. Find the bilinear transformation which maps z_1, z_2, z_3 to w_1, w_2, w_3 respectively where

(a) $z_1 = 2, z_2 = i, z_3 = -2; w_1 = 1, w_2 = i, w_3 = -1.$

(b) $z_1 = -i, z_2 = 0, z_3 = i; w_1 = -1, w_2 = i, w_3 = 1.$

(c) $z_1 = \infty, z_2 = i, z_3 = 0; w_1 = 0, w_2 = i, w_3 = \infty.$

(d) $z_1 = 1, z_2 = -i, z_3 = 0; w_1 = 0, w_2 = 2, w_3 = -i.$

2. Find the bilinear transformation which maps the points z_1, z_2, z_3 into the points $w_1 = 0, w_2 = 1$ and $w_3 = \infty$.

3. Find a bilinear transformation which maps the vertices $1+i, -i, 2-i,$ of a triangle of the z-plane into the points $0, 1, i$ of the w-plane.

Answers: 1. (a) $w=\dfrac{3z+2i}{iz+6}.$ **)(b)** $w=\dfrac{i(1-z)}{z+1}.$ **(c)** $w=\dfrac{-1}{z}.$

(d) $w=\dfrac{2(z-1)}{1+iz-2}.$

2. $w=\dfrac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}.$

3. $w=\dfrac{z-(2+2i)}{(i-1)z(3+5i)}.$

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7.4 FIXED POINTS OF BILINEAR TRANSFORMATIONS

Definition 7.4.1

If $w = f(z)$ is any transformation from the z-plane to w-plane, the ***fixed points*** of the transformation are the solutions of the equation $z = f(z)$.

Consider a bilinear transformation given by

$$w = \frac{az+b}{cz+d} \text{ where } ad - bc \neq 0.$$

The ***fixed points*** or ***invariant points*** of the bilinear transformation are given by the roots of the equation $z = \frac{az+b}{cz+d}$.

$$(i.e)\ cz^2 + (d-a)z - b = 0.$$

Case(i) $c \neq 0.$

In this case the fixed points are given by

$$z = \frac{(a-d) \pm \sqrt{[(d-a)^2 + 4bc}}{2c}$$

When $(d-a)^2 + 4bc \neq 0,$ the given bilinear transformation has two finite fixed points and when $(d-a)^2 + 4bc = 0$ it has only one finite fixed point.

Case(ii) $c = 0.$

In this case, the bilinear transformation becomes $w = \left(\frac{a}{d}\right)z + \frac{b}{d}$.

Clearly ∞ is one fixed point.

Other fixed point is determined by the equation $z = \left(\frac{a}{d}\right)z + \frac{b}{d}$.

(i.e) $(d-a)z - b = 0$.

$\therefore$ If $d - a \neq 0$,we get a finite fixed point $\frac{b}{d-a}$.

Thus we have

Case (i) $c \neq 0;\ (d-a)^2 + 4bc \neq 0 \Rightarrow$ 2 finite fixed points.

Case (ii) $c \neq 0;\ (d-a)^2 + 4bc = 0 \Rightarrow$ one finite fixed point.

Case (iii) $c = 0;\ a \neq d \Rightarrow \infty$ and one finite fixed point.

Case (iv) $c = 0;\ a = d \Rightarrow \infty$ is the only fixed point.

Theorem 7.4.2

Any bilinear transformation having two finite fixed points α and β can be written in the form $\frac{w-\alpha}{w-\beta} = k\left(\frac{z-\alpha}{z-\beta}\right)$.

Proof.

Let T be the given bilinear transformation having α and β as fixed points. Let the image of any point γ under T be δ.

Then the bilinear transformation T is given by $(w, \delta, \alpha, \beta) = (z, \gamma, \alpha, \beta)$.

$$\therefore \frac{(w-\alpha)(\delta-\beta)}{(w-\beta)(\delta-\alpha)} = \frac{(z-\alpha)(\gamma-\beta)}{(z-\beta)(\gamma-\alpha)}.$$

(i.e) $\frac{w-\alpha}{w-\beta} = k\left(\frac{z-\alpha}{z-\beta}\right)$ where $k = \frac{(\gamma-\beta)(\delta-\alpha)}{(\gamma-\alpha)(\delta-\beta)}$. ....(1)

Definition 7.4.3

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Let T be a bilinear transformation with two finite fixed points α, β. If k given by (1) is real T is called ***hyperbolic*** and if $|k| = 1$, T is called ***elliptic.***

Theorem 7.4.4

Any bilinear transformation having ∞ and $\alpha \neq \infty$ as fixed points can be written in the form $w - \alpha = k(z - \alpha)$.

Proof.

Let T be the given bilinear transformation having ∞ and α as fixed points. Let the image of any point γ under T be δ.

Then the bilinear transformation is given by $(w, \delta, \alpha, \infty) = (z, \gamma, \alpha, \infty)$.

$$\therefore \frac{w-\alpha}{\delta-\alpha} = \frac{z-\alpha}{\gamma-\alpha}.$$

$$\therefore w - \alpha = k(z - \alpha) \quad \text{where } k = \frac{\delta-\alpha}{\gamma-\alpha}.$$

Definition 7.4.5

A bilinear transformation with only one finite fixed point is called ***parabolic.***

Theorem 7.4.6

Any bilinear transformation having ∞ as the only fixed point is a translation.

Proof.

Let $w = \dfrac{az+b}{cz+d}$ be the bilinear transformation having ∞ as the only fixed point.

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Then $c = 0$ and $a = d$.

$\therefore$ The bilinear transformation reduces to the form $w = \dfrac{az+b}{a}$.

$\therefore w = z + \left(\dfrac{b}{a}\right)$ which is a translation.

Theorem 7.4.7

Let C be a circle or a straight line and z_1, z_2 be inverse points or reflection points with respect to C. Let w_1, w_2 and C_1 be the images of z_1, z_2 and C under a bilinear transformation. Then w_1 and w_2 are inverse points or reflection points with respect to C_1.

(i.e) A *bilinear transformation preserves inverse points.*

Proof.

Let the equation of C be $pz\bar{z} + \alpha\bar{z} + \bar{\alpha}z + \beta = 0$. .....(1)

Result: z_1 and z_2 are inverse points with respect to the circle $z\bar{z} + \bar{\alpha}z + \alpha\bar{z} + \beta = 0$ iff $z_1\bar{z}_2 = \bar{\alpha}z_1 + \alpha\bar{z}_2 + \beta = 0$.

Since z_1 and z_2 are inverse points w.r.t. C by the above result, we have

$$pz_1\overline{z_2} + \alpha\overline{z_2} + \overline{\alpha}z_1 + \beta = 0. \quad .....(2)$$

Let the given bilinear transformation be $w = \dfrac{az+b}{cz+d}$ where $ad - bc \neq 0$.

$$\therefore \quad z = \frac{dw-b}{-cw+a}.$$

$\therefore$ Under the given bilinear transformation (1) is transformed into

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$$p\left[\frac{dw-b}{-cw+a}\right]\left[\frac{\overline{d}\,\overline{w}-\overline{b}}{-\overline{c}\,\overline{w}+\overline{a}}\right]+\alpha\left[\frac{\overline{d}\,\overline{w}-\overline{\beta}}{-\overline{c}\,\overline{w}+\overline{a}}\right]+\overline{\alpha}\left[\frac{dw-b}{-cw+a}\right]+\beta=0$$

……(3)

Also (2) is transformed into

$$p\left[\frac{dw_1-b}{-cw_1+a}\right]\left[\frac{\overline{d}\,\overline{w_2}-\overline{b}}{-\overline{c}\,\overline{w_2}+\overline{a}}\right]+\alpha\left[\frac{\overline{d}\,\overline{w_2}-\overline{\beta}}{-\overline{c}\,\overline{w_2}+\overline{a}}\right]+\overline{\alpha}\left[\frac{dw_1-b}{-cw_1+a}\right]+\beta=0$$

…..(4)

Clearly (4) is the condition for w_1 and w_2 to be inverse points with respect to (3). Hence the theorem.

Solved Problems

Problem 7.4.8

Find the invariant points of the transformations

(a) $w=\dfrac{1+z}{1-z}$ (b) $w=\dfrac{1}{z-2i}$

Solution.

(a) The invariant points of $w=f(z)$ are got from $f(z)=z$.

$$\therefore f(z)=z \Rightarrow z=\frac{1+z}{1-z}$$

$$\Rightarrow z-z^2=1+z$$

$$\Rightarrow 1+z^2=0.$$

$$\Rightarrow z=\pm i.$$

$\therefore$ i and $-i$ are the two fixed points of the transformation.

(b) $f(z)=z \Rightarrow z=\dfrac{1}{z-2i}$

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$\Rightarrow z^2 - 2iz - 1 = 0.$

$\Rightarrow (z - i)^2 = 0.$

Hence i is the (only) fixed point.

Problem 7.4.9

Prove that the transformation $w = \bar{z}$ is not a bilinear transformation.

Solution.

Any bilinear transformation, other than the identity transformation has two fixed points. However the transformation $w = \bar{z}$ has infinitely many fixed points, namely all real numbers. Hence it is not a bilinear transformation.

Note 7.4.10

The above result can also be established by showing the $w = \bar{z}$ does not preserve cross ratio.

Exercises 7.4.11

1. Prove that a bilinear transformation having origin as the fixed point can be written in the form $w = \dfrac{z}{cz+d}$.

2. Prove that a bilinear transformation having 0 and ∞ as fixed point is of the form $w = az$.

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3. Find the fixed points and normal form of the following bilinear transformations. Also determine whether they are elliptic, hyperbolic or parabolic.

(i) $w = z + 3$ (ii) $w = 2z + 3$.

(iii) $w = \frac{z-1}{z+1}$

(iv) $w = 6z - 9z$

(v) $w = \frac{z}{2-z}$ (vi) $w = \frac{3z-4}{z-1}$

Answers:3.(i)∞ (ii)∞ (iii) $\pm i; \frac{w-i}{w+i} = \frac{i-1}{i+1} \cdot \frac{z-i}{z+i}$;*elliptic* (iv)3;parabolic

(v)0,1; $\frac{w}{w-1} = \frac{1}{2} \cdot \frac{z}{z-1}$; *hyperbolic* (vi)2; parabolic

7.5 SOME SPECIAL BILINEAR TRANSFORMATIONS

In this section, we shall determine the general form of the transformations which map

(i) the real axis onto itself.

(ii) the unit circle onto itself.

(iii) the real axis onto the unit circle.

Theorem 7.5.1

A bilinear transformation $w = \frac{az+b}{cz+d}$ where $ad - bc \neq 0$ maps the real axis into itself if and only if a,b,c,d are real.

Further this transformation maps the upper half plane $Im\ z \geq 0$

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into the upper half plane

$Im\, w \geq 0$ if and only if $ad - bc > 0$.

Proof.

Suppose a, b, c, d are real.

Then obviously, z is real $\Rightarrow$ w is also real.

Therefore the real axis is mapped into itself.

Conversely consider any bilinear transformation T that maps the real axis into itself.

Therefore there exists real numbers x_1, x_2, x_3 such that

$T(x_1) = 1,\ T(x_2) = 0$ and $T(x_3) = \infty$

Therefore bilinear transformation T is given by

$(z, x_1, x_2, x_3) = (w, 1, 0, \infty)$

$$\therefore \frac{(z - x_2)(x_1 - x_3)}{(z - x_3)(x_1 - x_2)} = \frac{w - 0}{1 - 0} = w.$$

$$\therefore \frac{az + b}{cz + d} = w$$

where $a = x_1 - x_3$; $b = -x_2(x_1 - x_3)$; $c = x_1 - x_2$ and $d = -x_3(x_1 - x_2)$.

Since x_1, x_2, x_3 are real, a, b, c are also real.

Now,

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$$2i \text{ Im } w = w - \bar{w} = \frac{az+b}{cz+d} - \frac{a\bar{z}+b}{c\bar{z}+d}$$

$$= \frac{(ad-bc)(z-\bar{z})}{|cz+d|^2}$$

$$= 2i\left(\frac{ad-bc}{|cz+d|^2}\right)\text{Im } z$$

$$\therefore \text{Im } w = \frac{(ad-bc)}{|cz+d|^2}\text{Im } z.$$

Therefore, the upper half plane $Im\ z \geq 0$ is mapped onto the upper half plane $Im\ w \geq 0 \Leftrightarrow$ $bc > 0$.

Theorem 7.5.2

Any bilinear transformation which maps the unit circle $|z| = 1$ into the unit circle $|w| = 1$ can be written in the form $w = e^{i\lambda}\left(\frac{z-\alpha}{\bar{\alpha}z-1}\right)$

where λ is real.

Further this transformation maps the circular disc $|z| \leq 1$ onto the circular disc $|w| \leq 1$ iff $|\alpha| < 1$.

Proof.

Let $w = \frac{az+b}{cz+d}$ where $ad - bc \neq 0$ be any bilinear transformation which maps $|z| = 1$ on to $|w| = 1$.

0 and ∞ are inverse points with respect to the circle $|w| = 1$.

Hence their pre-images $-(b/a)$ and $-(d/c)$ are inverse points with respect to $|z| = 1$.

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Result: *z_1 and z_2* **are inverse points with respect to the circle** $z\bar{z} + \bar{\alpha}z + \alpha\bar{z} + \beta = 0$ **iff** $z_1\bar{z}_2 = \bar{\alpha}z_1 + \alpha\bar{z}_2 + \beta = 0$.

Therefore $(-b/a)(\bar{d}/c) = 1$ (using above result)

Therefore if $\alpha = -(b/a)$, then $1/\bar{\alpha} = -d/c$.

Therefore $$w = \frac{az+b}{cz+d}$$

$$= \left(\frac{a}{c}\right)\left[\frac{z-\alpha}{z-(1/\bar{\alpha})}\right]$$

$$= \left(\frac{a\bar{\alpha}}{c}\right)\left(\frac{z-\alpha}{\bar{\alpha}z-1}\right)$$

Now, let $|z| = 1$. Hence $|w| = 1$.

Therefore $$1 = |w| = \left|\frac{a\bar{\alpha}}{c}\right|\left|\frac{\bar{z}-\bar{\alpha}}{\bar{\alpha}z-z\bar{z}}\right| \qquad (\text{since } z\bar{z} = 1)$$

$$= \left|\frac{a\bar{\alpha}}{c}\right|\left|\frac{\bar{z}-\bar{\alpha}}{\bar{\alpha}-\bar{z}}\right|$$

$$= \left|\frac{a\bar{\alpha}}{c}\right|$$

Thus $$\left|\frac{a\bar{\alpha}}{c}\right| = 1$$

$$\therefore \quad \frac{a\bar{\alpha}}{c} = e^{i\lambda}$$ for some real number λ.

$$\therefore w = e^{i\lambda}\left(\frac{z-\alpha}{\bar{\alpha}z-1}\right)$$ where λ is real

Now, $$w\bar{w} - 1 = e^{i\lambda}\left(\frac{z-\alpha}{\bar{\alpha}z-1}\right)e^{-i\lambda}\left(\frac{\bar{z}-\bar{\alpha}}{\alpha\bar{z}-1}\right) - 1$$

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$$= \frac{(z-\alpha)(\bar{z}-\bar{\alpha})}{(\bar{\alpha}z-1)(\alpha\bar{z}-1)} - 1$$

$$= \frac{(1-\alpha\bar{\alpha})(z\bar{z}-1)}{|\alpha\bar{z}-1|^2}$$

$\therefore$ The transformation maps $|z| \le 1$ onto $|w| \le 1$

$$\Leftrightarrow 1-\alpha\bar{\alpha} > 0$$

$$\Leftrightarrow \alpha\bar{\alpha} < 1$$

$$\Leftrightarrow |\alpha| < 1.$$

Theorem 7.5.3

Any bilinear transformation which maps the real axis onto unit circle $|w| = 1$ can be written in the form $w = e^{i\lambda}\left(\frac{z-\alpha}{z-\bar{\alpha}}\right)$ where λ is real.

Further this transformation maps the upper half plane $Im\, z \ge 0$ onto the unit circular disc $|w| \le 1$ iff $Im\, \alpha > 0$.

Proof.

$w = \frac{az+b}{cz+d}$ where $ad - bc \ne 0$ be any bilinear transformation which maps real axis onto the unit circle $|w| = 1$.

0 and ∞ are inverse points with respect to the unit circle $|w| = 1$.
Hence their pre-images $-(b/a)$ and $-(d/c)$ are reflection points with respect to the real axis.

$$\therefore \text{ If } \alpha = -\left(\frac{b}{a}\right), \quad \textit{then} \quad \bar{\alpha} = -\left(\frac{d}{c}\right)$$

Now, $$w = \frac{az+b}{cz+d}$$

$$= \left(\frac{a}{c}\right)\left[\frac{z+(b/a)}{z+(d/c)}\right]$$

$$= \left(\frac{a}{c}\right)\left[\frac{z-\alpha}{z-\bar{\alpha}}\right]$$

Now, suppose z is real. Hence $|w| = 1$.

$$\therefore \left|\frac{a}{c}\right|\frac{|z-\alpha|}{|z-\bar{\alpha}|} = 1.$$

Now, since z is real $z = \bar{z}$ and hence

$|z-\alpha| = |\overline{z-\alpha}| = |\bar{z}-\bar{\alpha}| = |z-\bar{\alpha}|$

$\therefore \left|\frac{a}{c}\right| = 1$. Hence $\frac{a}{c} = e^{i\lambda}$ where λ is real.

$\therefore w = e^{i\lambda}\left(\frac{z-\alpha}{z-\bar{\alpha}}\right)$, where λ is real, is the required transformation.

Now, $w\bar{w}-1 = e^{i\lambda}\left(\frac{z-\alpha}{z-\bar{\alpha}}\right)e^{-i\lambda}\left(\frac{\bar{z}-\bar{\alpha}}{\bar{z}-\alpha}\right)-1$

$= \left(\frac{z-\alpha}{z-\bar{\alpha}}\right)\left(\frac{\bar{z}-\bar{\alpha}}{\bar{z}-\alpha}\right)-1$

$= \frac{-4\,\text{Im}\,z\ \text{Im}\,\alpha}{|z-\bar{\alpha}|^2}$

$\therefore$ The bilinear transformation maps the upper half plane $Im\ z \geq 0$ onto the disc $|w| \leq 1$ iff $Im\ \alpha > 0$.

Solved Problems

Problem 7.5.4

Find the general bilinear transformation which maps the unit circle $|z|=1$ onto $|w| = 1$ and the points $z = 1$ to $w = 1$ and $z = -1$ to $w = -1$.

Solution.

We know any bilinear transformation which maps $|z| = 1$ onto $|w| = 1$ is of the form $w = e^{i\lambda}\left(\frac{z-\alpha}{\bar{\alpha}z-1}\right)$, where λ is real.

Since 1 and -1 are again mapped to 1, -1 respectively, we have

$1 = e^{i\lambda}\left(\frac{1-\alpha}{\bar{\alpha}-1}\right)$ ....(1)

$-1 = e^{i\lambda}\left(\frac{-1-\alpha}{-\bar{\alpha}-1}\right) = e^{i\lambda}\left(\frac{1+\alpha}{1+\bar{\alpha}}\right)$ ....(2)

Dividing (1) by (2) we get, $-1 = \left(\frac{1-\alpha}{\bar{\alpha}-1}\right)\left(\frac{1+\bar{\alpha}}{1+\alpha}\right)$

$\therefore\ -\bar{\alpha}-\alpha\bar{\alpha}+1+\alpha = 1+\bar{\alpha}-\alpha-\alpha\bar{\alpha}$.

$\therefore\ -2\bar{\alpha}+2\alpha = 0$.

$\therefore\ \alpha = \bar{\alpha}$ ....(3)

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Using (3) in (1), we get, $1 = -e^{i\lambda}$.
Hence $e^{i\lambda} = -1$.

$\therefore$

The required is $w = \dfrac{\alpha - z}{\alpha z - 1}$.

Problem 7.5.5

Prove that the transformation given by $\bar{a}wz - bw - \bar{b}z + a = 0$ maps the unit circle $|z| = 1$ onto transformation the unit circle $|w| = 1$ if $|b| \neq |a|$.

Solution.

$$\bar{a}wz - bw - \bar{b}z + a = 0.$$

$$\therefore w = \frac{\bar{b}z - a}{\bar{a}z - b}.$$

$$w\bar{w} - 1 = \left(\frac{\bar{b}z - a}{\bar{a}z - b}\right)\left(\frac{b\bar{z} - \bar{a}}{a\bar{z} - \bar{b}}\right) - 1$$

$$= \frac{(z\bar{z} - 1)(|b|^2 - |a|^2)}{|\bar{a}z - b|^2}$$

$$If\ |b| \neq |a|, then\ w\bar{w} - 1 = 0 \Leftrightarrow z\bar{z} - 1 = 0$$

Therefore the unit circle $|z| = 1$ is mapped onto the unit circle $|w| = 1$ if $|b| \neq |a|$.

Problem 7.5.6

Show that the bilinear transformation which maps the unit circle $|z| = 1$ onto the unit circle $|w| = 1$ can be put in the form

$$w = e^{i\lambda}\left(\frac{az + b}{\bar{b}z + \bar{a}}\right) \quad \text{where } \lambda \text{ is real.}$$

Further this transformation maps the circular disc $|z| \leq 1$ onto the circular disc
$|w| \leq 1$ iff $|a| > |b|$.
Also if the point $z = 1$ is the only invariant point, show that the transform

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ation may written as $\frac{1}{w-1} = \frac{1}{z-1} + \frac{1}{k}$ where $k = 1 + \frac{\bar{a}}{\bar{b}}$.

Solution.

We know that any bilinear transformation which maps $|z| = 1$ onto $|w| = 1$ can be written in the form $w = e^{i\mu}\left(\frac{z-\alpha}{\bar{\alpha}z-1}\right)$ where μ is real and this maps $|z| \le 1$ onto $|w| \le 1$ iff $|\alpha| < 1$.

Now, choose $a = 1$ and $b = -\alpha$.

$$\therefore w = e^{i\mu}\left(\frac{z-\alpha}{\bar{\alpha}z-1}\right)$$

$$= e^{i\mu}\left(\frac{az+b}{-\bar{b}z-\bar{a}}\right) = -e^{i\mu}\left(\frac{az+b}{\bar{b}z+\bar{a}}\right)$$

$$= e^{i\lambda}\left(\frac{az+b}{\bar{b}z+\bar{a}}\right) \textit{ where } e^{i\lambda} = -e^{i\mu} \textit{ and } \lambda \textit{ is real} \ldots\ldots\ldots\ldots(1)$$

Further $|\alpha| < 1 \Leftrightarrow |-b| < a$ (since $a = 1$)

$\Leftrightarrow |b| < |a|$.

The transformation (1) maps $|z| \le 1$ onto $|w| \le 1$ iff $|a| > |b|$.

Now, suppose $z = 1$ is the only fixed point of (1).

Therefore $z=1$ is the only root of the equation $z = e^{i\lambda}\left(\frac{az+b}{\bar{b}z+\bar{a}}\right)$

$(i.e)\ \bar{b}z^2 + (\bar{a} - ae^{i\lambda})z - be^{i\lambda} = \bar{b}(z-1)^2$.

Equation the corresponding coefficients, we get,

$\bar{a} - ae^{i\lambda} = -2\bar{b}.$ ....(2)

$be^{i\lambda} = -\bar{b}$ ....(3)

(2) *can be written as* $\bar{a} + \bar{b} = -\bar{b} + ae^{i\lambda}$ ....(4)

Using (3) *we get* $\bar{a} - be^{i\lambda} = ae^{i\lambda} - \bar{b}$ ....(5)

Now,

$$w - 1 = e^{i\lambda}\left(\frac{az+b}{\bar{b}z+\bar{a}}\right) - 1$$

$$= \frac{e^{i\lambda}az + be^{i\lambda} - \bar{b}z - \bar{a}}{\bar{b}z + \bar{a}}$$

$$= \frac{(ae^{i\lambda} - \bar{b})z + (be^{i\lambda} - \bar{a})}{\bar{b}z + \bar{a}}$$

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$$= \frac{(ae^{i\lambda} - \bar{b})z - (\bar{a} - be^{i\lambda})}{\bar{b}z + \bar{a}} = \frac{(z-1)(ae^{i\lambda} - \bar{b})}{\bar{b}z + \bar{a}} \qquad (using\ 5)$$

$$= \frac{(z-1)(ae^{i\lambda} - \bar{b})}{\bar{b} + \bar{a} + (z-1)\bar{b}} = \frac{(z-1)(\bar{a} + \bar{b})}{(\bar{a} + \bar{b}) + (z-1)\bar{b}} \qquad (using\ 4)$$

$$\therefore \frac{1}{w-1} = \frac{(\bar{a} + \bar{b}) + (z-1)\bar{b}}{(z-1)(\bar{a} + \bar{b})} = \frac{1}{z-1} + \frac{\bar{b}}{\bar{a} + \bar{b}}$$

$$= \frac{1}{z-1} + \frac{1}{1 + (\bar{a}/\bar{b})} = \frac{1}{z-1} + \frac{1}{k} \qquad where\ k = 1 + \frac{\bar{a}}{\bar{b}}.$$

Exercises 7.5.7

1. Prove that any bilinear transformation which maps the imaginary axis onto the unit circle $|w|=1$ can be written in the form $w = e^{i\lambda}\left(\frac{z-\alpha}{z+\alpha}\right)$. Further this transformation maps the upper half plane $Re\ z \geq 0$ onto the unit circular disc $|w| \leq 1$ iff $Re\ \alpha > 0$.

2. Show that the bilinear transformation $w = \left(\frac{1+z}{1-z}\right)$ maps the region $|z| \leq 1$ onto the half plane $Re\ w \geq 0$.

UNIT 8

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8.1 CAUCHY'S FUNDAMENTAL THEOREM

In this section, we prove the fundamental theorem of integration known as Cauchy's theorem which forms the basis for the theory of complex integration.

Definition 8.1.1

Let $p(x,y)$ and $q(x,y)$ be two real valued functions. Then the differential equation $p(x,y)\,dx + q(x,y)\,dy = 0$ is said to be ***exact*** if there exists a function $u(x,y)$ such that $\frac{\partial u}{\partial x} = p$ *and* $\frac{\partial u}{\partial y} = q$.

We assume the following theorem without proof.

Theorem 8.1.2

$\int_C pdx + qdy$ depends only on the end points of C if and only if the integrand is exact.

Remark 8.1.3

The above theorem is true if p and q are complex valued functions as well.

We now apply the above theorem for complex functions to get a characterisation for $\int_C f(z)dz$ to depend only on end points of C.

Theorem 8.1.4

Let $f(z)$ be a continuous complex valued function defined on a region D. Then $\int_C f(z)dz$ depends only on the end points of C if and only if there exists an analytic function $F(z)$ such that $F'(z) = f(z)$ in D.

Proof.

$$\int_C f(z)dz = \int_C f(z)(dx + idy) \quad (\text{since } z = x+iy)$$

$$= \int_C f(z)dx + if(z)dy$$

$\int_C f(z)dz$ depends only on the end points of C if and only if there exists a function $F(z)$ defined on D such that $\frac{\partial F}{\partial x} = f(z)$ *and* $\frac{\partial F}{\partial y} = if(z)$.

Therefore, $\frac{\partial F}{\partial x} = \frac{1}{i}\frac{\partial F}{\partial y}$ *so that* $\frac{\partial F}{\partial x} = -i\frac{\partial F}{\partial y}$ which is the complex form of the Cauchy-Riemann equation for $F(z)$.

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Since $f(z)$ is continuous, the partial derivatives of $F(z)$ are also continuous and hence $F(z)$ is analytic in D, $F'(z) = f(z)$. Hence the theorem.

Corollary 8.1.5

Let $f(z)$ be a continuous complex valued function defined on a region D, then $\int_C f(z)dz = 0$ for every closed curve C lying in D iff there exists an analytic function $F(z)$ such that $F'(z) = f(z)$ in D.

Corollary 8.1.6

$\int_C (z-a)^n dz = 0$ for every closed curve C provided $n \geq 0$.

Proof. Let $F(z) = \dfrac{(z-a)^{n+1}}{n+1}$.

Clearly $F'(z) = (z-a)^n$

$= f(z)$.

$\therefore$ By corollary 8.1.5, $\int_C f(z)dz = 0$. Hence $\int_C (z-a)^n dz = 0$ *for ll* $n \geq 0$.

Lemma 8.1.7

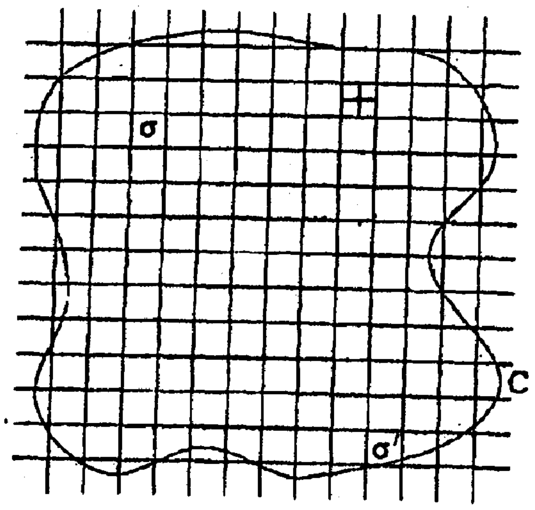
Let C be a simple closed curve. Let D denote the closed region consisting of all points interior to C together with the points on C. Let f be a function analytic in D. Then given $\varepsilon > 0$,it is possible to cover D with a finite number of squares and partial squares whose boundaries are denoted by C_j such that there exist points z_j lying inside or on each C_j satisfying

$$\left|\frac{f(z)-f(z_j)}{z-z_j} - f'(z_j)\right| < \varepsilon \quad (j = 1,2,\ldots,n) \quad \ldots\ldots(1)$$

For all points z distinct from each z_j and lying inside or on C_j.

Proof.

We subdivide the region D into squares and partial squares by drawing equally spaced lines parallel to the coordinate axes (refer figure). (A square is a closed region consisting of all points on and interior to it. If a particular square contains points which are not in D, we remove those points and call what remains a partial square. In this figure σ is a square and σ^1 is a partial square). This gives a finite number of squares and partial squares which cover the region D.



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Suppose the Lemma is false. Then in the covering constructed as above, there exists a subregion with boundary C_j such that no point z_j exists satisfying (1).

Let σ_0 denote that subregion if it is a square. If it is a partial square let σ_0 denote the entire square of which it is a part.

We now subdivide σ_0 into four smaller squares by drawing line segments joining the mid points of the opposite sides. At least one of the four smaller squares say σ_1 is such that σ_1 contains points of D and no point z_j satisfying (1) exists.

Continuing this process, we obtain a nested infinite sequence of squares $\sigma_1, \sigma_2, \ldots, \sigma_n, \ldots$ such that for each σ_n, no z_j satisfying (1) exists.

Now there exists a point z_0 common to each σ_n such that for any $\delta > 0$ the neighbourhood $|z - z0| < \delta$ contains all the squares σ_n for all sufficiently large values of n.

Hence every neighbourhood of z_0 contains points of D distinct from z_0. Hence z_0 is a limit point of D. Since D is closed $z_0 \in D$.
Since $f(z)$ is analytic at z_0 there exists $\delta > 0$ such that

$$|z - z_0| < \delta \Rightarrow \left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| < \varepsilon \qquad \ldots\ldots(2)$$

Choose N such that the square σ_N is contained in the neighbourhood $|z - z_0| < \delta$.
Then for every point z in σ_N (2) holds.
Therefore z_0 serves as the point z_j stated in the lemma. This is a contradiction since there is no z_j in σ_N satisfying (1).

This contradiction proves the lemma.

Theorem 8.1.8 (Cauchy's Theorem)
Let f be a function which is analytic at all points inside and on a simple closed curve C. Then $\int_C f(z)dz = 0$.

Proof. Let D be the closed region consisting of all points interior to C together with the points on C.
Let $\varepsilon > 0$ be given.
Let C_j $(j = 1, 2, \ldots, n)$ denote the boundaries of the squares and partial squares covering D such that there exists a point z_j lying inside or on C_j satisfying

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$$\left|\frac{f(z)-f(z_j)}{z-z_j}-f'(z_j)\right|<\varepsilon \qquad \text{.......(1)}$$

for all z distinct from z_j and lying within or on C_j.

Let $$\delta_j(z) = \begin{cases} \dfrac{f(z)-f(z_j)}{z-z_j}-f'(z_j) & \text{if } z \neq z_j \\ 0 & \text{if } z = z_j \end{cases}$$

Clearly $\delta_j(z)$ is a continuous function and

$$f(z) = f(z_j) - z_j f'(z_j) + z f'(z_j) + (z-z_j)\delta_j(z).$$

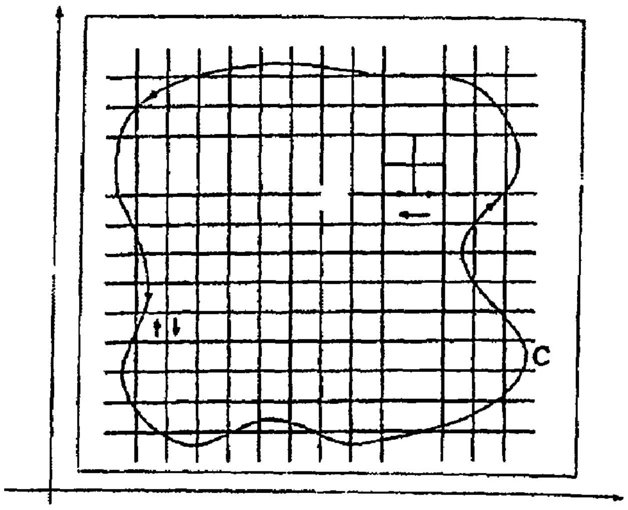
$$\therefore \int_{C_j} f(z)dz = \int_{C_j} f(z_j)dz - \int_{C_j} z_j f'(z_j)dz + \int_{C_j} z f'(z_j)dz + \int_{C_j}(z-z_j)\delta_j(z)dz$$

$$= f(z_j)\int_{C_j} dz - z_j f'(z_j)\int_{C_j} dz + f'(z_j)\int_{C_j} zdz + \int_{C_j}(z-z_j)\delta_j(z)dz$$

$$= \int_{C_j}(z-z_j)\delta_j(z)dz \quad (\text{since} \int_{C_j} dz = 0 \text{ and } \int_{C_j} zdz = 0)$$

$$\therefore \sum_{j=1}^{n}\int_{C_j} f(z)dz = \sum_{j=1}^{n}\int_{C_j}(z-z_j)\delta_j(z)dz \qquad \text{.......(2)}$$

Now, in the sum $\sum_{j=1}^{n}\int_{C_j} f(z)dz$ the integrals along the common boundary of every pair of adjacent subregions cancel each other. (since the integral is taken in one direction along that line segment in one subregion and in the opposite direction in the other) *(refer figure)*



Hence only the integrals along the arcs which are the parts of C remain.

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$$\therefore \sum_{j=1}^{n} \int_{C_j} f(z)dz = \int_{C} f(z)dz$$

$\therefore$From (2), $\int_{C} f(z)dz = \sum_{j=1}^{n} \int_{C_j} (z-z_j)\delta_j(z)dz$.

$$\therefore \left|\int_{C} f(z)dz\right| = \left|\sum_{j=1}^{n} \int_{C_j} (z-z_j)\delta_j(z)dz\right|$$

$$\leq \sum_{j=1}^{n} \int_{C_j} |(z-z_j)\delta_j(z)|\, dz$$

$$= \sum_{j=1}^{n} \int_{C_j} |(z-z_j)||\delta_j(z)|\, dz$$

$$\therefore \left|\int_{C} f(z)dz\right| \leq \sum_{j=1}^{n} \int_{C_j} |(z-z_j)||\delta_j(z)|\, dz \qquad ....(3)$$

Now if C_j is a square and s_j is the length of its side, then $|z - z_j| < \sqrt{2}\, s_j$ for all z on C_j

Also from (1), we have $|\delta_j(z)| < \varepsilon$ and hence

$\int_{C_j} |z-z_j||\delta_j(z)|\, dz < (\sqrt{2}\, s_j\varepsilon)(4s_j)$ (By Theorem

$\left|\int_{C} f(z)dz\right| \leq Ml$ *where* $M = \max\{|f(z)| : z \in C\}$)

$$= 4\sqrt{2}\, A_j\varepsilon. \qquad ....(4)$$

where A_j is the area of the square C_j.

Similarly for a partial square with boundary C_j if l_j is the length of the arc of C which forms a part of C_j., We have

$$\int_{C_j} |z-z_j||\delta_j(z)|\, dz < \sqrt{2}s_j\varepsilon(4s_j + l_j)$$

$$< 4\sqrt{2}\, A_j\varepsilon + \sqrt{2}\, Sl_j. \qquad ....(5)$$

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Where S is the length of a side of some square containing the entire region D as well as all the squares originally used in covering D.

We observe that the sum of all A_j's that occur in the right hand side of (4) and (5) do not exceed S^2 and the sum of all the l_j's is equal to L(the length of C).

Using (4) and (5) in (3), we obtain

$$\left|\int_C f(z)dz\right| < (4\sqrt{2}S^2 + \sqrt{2}SL)\varepsilon$$

$$= k\varepsilon \quad \text{where } k = 4\sqrt{2}\,S^2 + \sqrt{2}\,SL \text{ is a constant.}$$

Thus $\left|\int_C f(z)dz\right| \not< k\varepsilon.$

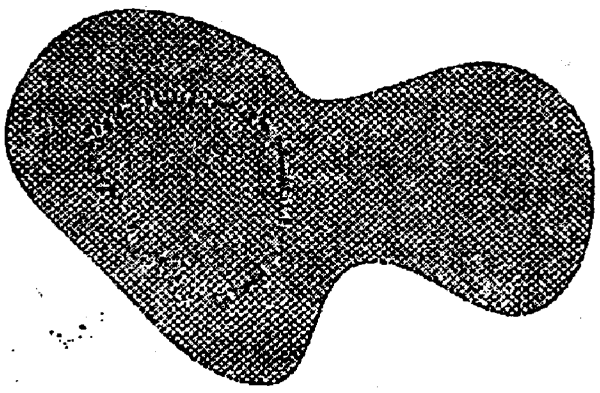
$$\int_C f(z)dz = 0, \text{since } \varepsilon \text{ is arbitrary}.$$

Note 8.1.9

Cauchy's theorem was first proved by using Green's theorem with the additional hypothesis that $f'(z)$ is continuous. Later , Goursat proved the theorem without the hypothesis that $f'(z)$ is continuous. For this reason the theorem is sometimes known as **Cauchy-Goursat theorem.**

Definition 8.1.10

A region D is said to be ***simply connected*** if every simple closed curve lying in D encloses only points of **D**.



For example, the interior of a simple closed curve is a simply connected region. The annular region enclosed by two concentric circles is not simply connected.
A region which is not a simply connected is said to be a multiply connected region.

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Intutively a simply connected region is one which does not have any holes in it.

We observe that Cauchy's theorem can be restated as follows.

Theorem 8.1.11 (Cauchy's Theorem for Simply Connected Regions)

Let f be a function which is analytic in a simply connected region D. Let C be any simple closed curve lying within D. Then $\int_C f(z)dz = 0$.

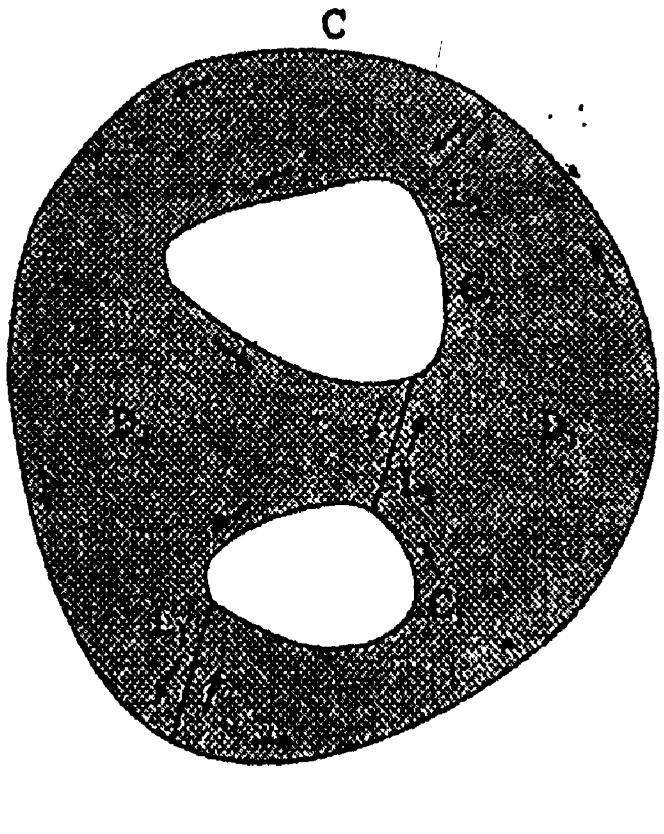
We now extend Cauchy's theorem to certain types of multiply connected regions.

Theorem 8.1.11 (Cauchy's Theorem for Multiply Connected Regions)

Let C be a simple closed curve. Let C_j ($j = 1, 2, ..., n$) be a finite number of simple closed curves lying in the interior of C such that the interiors of C_j's are disjoint. Let D be the closed region consisting of all points within and on C except the points interior to each C_j. Let B denote the entire oriented boundary of D consisting of C and all the C_j described in a direction such that the points of D are to the left of B. Let f be a function which is analytic in D. Then $\int_B f(z)dz = 0$.

Proof.

Let L_1 be a polygonal path joining a point of C to a point C_1; L_2 a polygonal path joining a point of C_1 to a point of C_2; ......; L_i a polygonal path joining a point of C_{i-1} to a point of C_i and L_{n+1} a polygonal path joining a oint of C_n to a point of C such that no two L_i's cross each other (refer figure).



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This divides the region D into two simply connected regions D_1 and D_2. Let B_1 and B_2 denote the boundaries of D_1 amd D_2 respectively.

By Cauchy's theorem for simply connected region,

$$\int_{B_1} f(z)dz = 0 \text{ and } \int_{B_2} f(z)dz = 0$$

Also . $\int_{B_1} f(z)dz + \int_{B_2} f(z)dz = \int_{B} f(z)dz$

Since the integrals along L_1 taken twice in the opposite directions and cancel each other.

$$\therefore \int_{B} f(z)dz = 0.$$

We observe that $B = C - C_1 - C_2 - \ldots\ldots - C_n$ and hence the above theorem can also be written in the form

$$\int_{C} f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz + \ldots\ldots + \int_{C_n} f(z)dz$$

In particular if C is a simple closed curve and C_0 is another simple closed curve lying in the interior of C and f is analytic in the region D consisting of all points inside and on C excluding the points interior to C_0 then,

$$\int_{C} f(z)dz = \int_{C_0} f(z)dz$$

8.2 CAUCHY'S INTEGRAL FORMULA

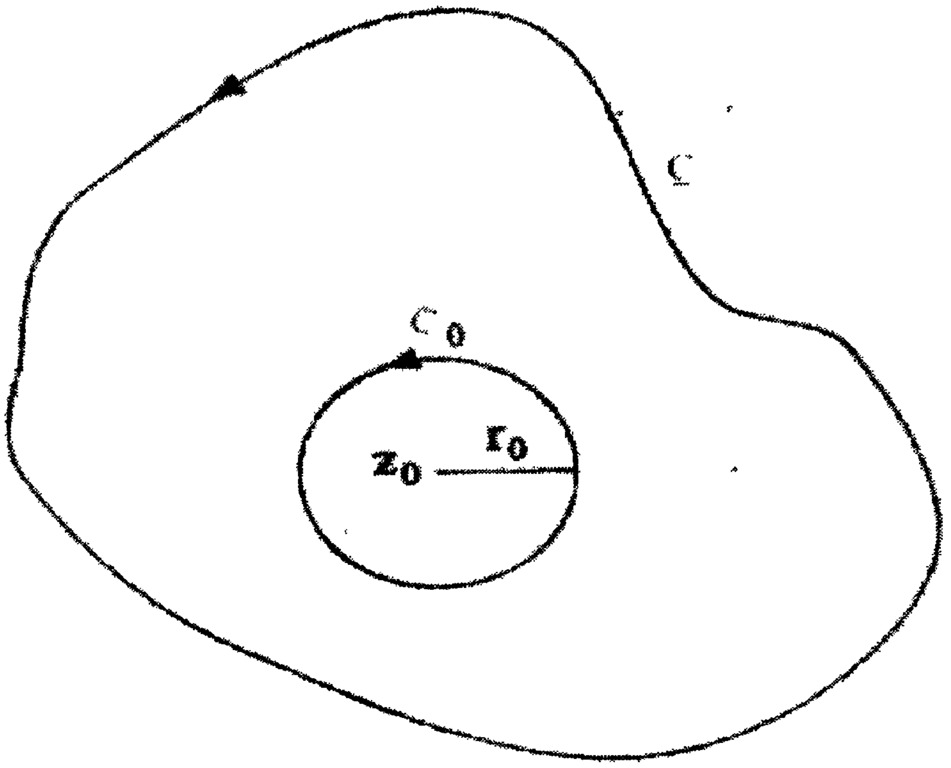
In this section we establish another fundamental result known as ***Cauchy's integral formula*** using Cauchy's theorem.

Theorem 8.2.1

Let $f(z)$ be a function which is analyatic inside and on a simple closed curve C. Let z_0 be any point in the interior of C.

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0} dz$$

Proof.



Choose a circle C_0 with centre z_0 and radius r_0 such that C_0 lies in the interior of C.

Now, z_0 is the only point inside C at which the function $\frac{f(z)}{z - z_0}$ is not analytic and hence is analytic in the region D consisting of all points inside and on C except the points interior to C_0.

Hence $\int_C \frac{f(z)dz}{z - z_0} = \int_{C_0} \frac{f(z)dz}{z - z_0}$

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$$= \int_{C_0} \left(\frac{f(z) - f(z_0) + f(z_0)}{z - z_0} \right) dz$$

$$= \int_{C_0} \left(\frac{f(z) - f(z_0)}{z - z_0} \right) dz + \int_{C_0} \frac{f(z_0)}{z - z_0} dz$$

$$= \int_{C_0} \left(\frac{f(z) - f(z_0)}{z - z_0} \right) dz + f(z_0) \int_{C_0} \frac{dz}{z - z_0}$$

$$= \int_{C_0} \left(\frac{f(z) - f(z_0)}{z - z_0} \right) dz + f(z_0)(2\pi i)$$

Thus $\int_C \frac{f(z)dz}{z - z_0} = \int_{C_0} \left(\frac{f(z) - f(z_0)}{z - z_0} \right) dz + 2\pi i f(z_0)$ ....(1)

We claim that $\int_{C_0} \left(\frac{f(z) - f(z_0)}{z - z_0} \right) = 0$

Since $f(z)$ is analytic inside and on C, it is continuous at z_0. Therefore ,given $\varepsilon > 0$, there exists $\delta > 0$ such that

$|z - z_0| < \delta$ implies $|f(z) - f(z_0)| < \varepsilon$

$$Hence \left| \int_{C_0} \left(\frac{f(z) - f(z_0)}{z - z_0} \right) dz \right| < \left(\frac{\varepsilon}{r_0} \right) (2\pi r_0)$$

(By Theorem $\left| \int_C f(z)dz \right| \leq Ml$ *where* $M = \max\{|f(z)| : z \in C\}$)

$$= 2\pi\varepsilon$$

$$\left| \int_{C_0} \left(\frac{f(z) - f(z_0)}{z - z_0} \right) dz \right| < 2\pi\varepsilon$$

Since ε is arbitrary, we have $\int_{C_0} \left(\frac{f(z) - f(z_0)}{z - z_0} \right) dz = 0$.

Therefore from (1), we get

$$\int_C \left(\frac{f(z)}{z-z_0}dz\right) = 2\pi i f(z_0).$$

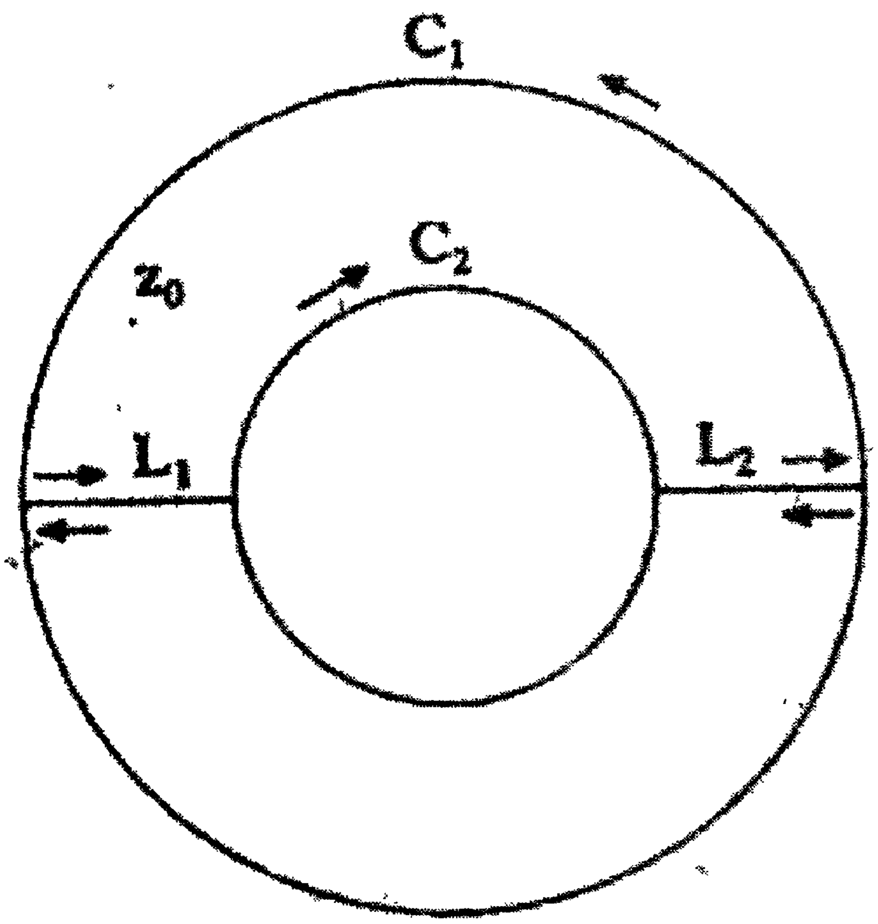
$$\therefore f(z_0) = \frac{1}{2\pi i}\int_C \left(\frac{f(z)}{z-z_0}dz\right)$$

Theorem 8.2.2

Let $f(z)$ be analytic in a region D bounded by two concentric circles C_1 and C_2 and on the boundary. Let z_0 be any point in D. Then

$$f(z_0) = \frac{1}{2\pi i}\int_{C_1} \frac{f(z)}{z-z_0} - \frac{1}{2\pi i}\int_{C_2} \frac{f(z)}{z-z_0}dz.$$

Proof.



Let L_1 and L_2 be two disjoint line segments not passing through z_0 both joining a point of C_1 to a point of C_2 as shown in the figure. This divides the region D into two simply connected regions D_1 and D_2. Let B_1 and B_2 denote the oriented boundary of D_1 and D_2 respectively.

Then $B_1 + B_2 = C_1 + C_2$.------------(1)

We assume without loss of generality that $z_0 \in D_1$.

By Cauchy's integral formula,

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$$f(z_0) = \frac{1}{2\pi i}\int_{B_1}\left(\frac{f(z)}{z - z_0}dz\right) \text{----------------------(2)}$$

Also $\frac{f(z)}{z - z_0}$ is analytic in D_2 and hence by Cauchy's theorem,

$$\frac{1}{2\pi i}\int_{B_2}\left(\frac{f(z)}{z - z_0}dz\right) = 0 \text{----------------------(3)}$$

Adding (2) and (3) and using (1), we get

$$f(z_0) = \frac{1}{2\pi i}\int_{C_1 - C_2}\left(\frac{f(z)}{z - z_0}dz\right)$$

$$f(z_0) = \frac{1}{2\pi i}\int_{C_1}\frac{f(z)}{z - z_0} - \frac{1}{2\pi i}\int_{C_2}\frac{f(z)}{z - z_0}dz.$$

Example 8.2.3

Consider $\int_C \frac{dz}{z-3}$ where C is the circle $|z-2| = 5$.

Let $f(z) = 1$.

The point $z = 3$ lies inside C.

Hence by Cauchy's integral formula, $\int_C \frac{dz}{z-3} = 2\pi i f(3) = 2\pi i$.

Example 8.2.4

Let C denote the unit circle $|z| = 1$.

$$\text{Then } \int_C \frac{e^z}{z}dz = \int_C \frac{e^z}{z-0}dz = 2\pi i e^0 = 2\pi i$$

Theorem 8.2.5

Let $f(z)$ be analytic inside and on the circle C with centre a and radius r. Then

$$f(a) = \frac{\int_0^l f(z)ds}{l}$$

where s is the arc length and l is the circumference of the circle.

(i.e) the value of the function at the centre is equal to the mean of the value of the function on the circumference.

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Proof.

By Cauchy's integral's formula , we have , $f(a)=\frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)}dz.$

Now the equation of the circle C is given by $z = a + r e^{i\theta}$ where $0 \le \theta \le 2\pi$.

$\therefore\ dz = ire^{i\theta}\ d\theta.$

$$\therefore\ f(a)=\frac{1}{2\pi i}\int_0^{2\pi}\frac{f(a+re^{i\theta})}{re^{i\theta}}(ire^{i\theta}d\theta)$$

$$=\frac{1}{2\pi}\int_0^{2\pi} f(a+re^{i\theta})d\theta$$

Also we have $s = r\theta$ and s varies from 0 to l.

$$\therefore\ d\theta=\frac{ds}{r}$$

$$\therefore\ f(a)=\frac{1}{2\pi r}\int_0^{l} f(a+re^{i\theta})ds$$

$$=\frac{1}{l}\int_0^{l} f(z)ds$$

Hence the theorem.

Theorem 8.2.6 (Maximum Modulus Theorem)

Let $f(z)$ be continuous in a closed and bounded region D and analytic and nonconstant in the interior of D. Then $|f(z)|$ attains its maximum value on the boundary of D and never in the interior of D.

Proof.

Since f is continuous in a closed and bounded region D, $|f(z)|$ is bounded and attains its bound.

Therefore there exists a positive real number M such that

$$|f(z)| \le M \text{ for all } z \in D \text{-----------------(1)}$$

and equality holds for at least one point z in D. Suppose that there exists an interior point $z_0 \in D$ such that

$$|f(z_0)| = M \text{---------------(2)}$$

Choose a circle with centre z_0 and radius r such that the circular disc $|z-z_0| \le r$ is contained in D. Then, we have

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$$f(z_0)=\frac{1}{2\pi}\int_0^{2\pi} f(z_0+re^{i\theta})e^{i\theta}d\theta, refer \quad proof\ of\ previous\ theorem$$

$$\therefore \quad f(z_0)\le\frac{1}{2\pi}\int_0^{2\pi}|f(z_0+re^{i\theta})|\,d\theta \text{----------(3)}$$

Also from (1) and (2), we *have* $|f(z_0 + r\,e^{i\theta})| \le |f(z_0)|$

$$\therefore \quad \int_0^{2\pi}|f(z_0+re^{i\theta})|\,d\theta\le 2\pi\,|f(z_0)|$$

$$\therefore |f(z_0)|\ge\frac{1}{2\pi}\int_0^{2\pi}|f(z_0+re^{i\theta})|\,d\theta \quad \text{-----------------(4)}$$

From (3) and (4), we get

$$|f(z_0)|=\frac{1}{2\pi}\int_0^{2\pi}|f(z_0+re^{i\theta})|\,d\theta$$

$$\therefore 2\pi\,|f(z_0)|=\int_0^{2\pi}|f(z_0+re^{i\theta})|\,d\theta$$

$$\int_0^{2\pi}|f(z_0)|\,d\theta=\int_0^{2\pi}|f(z_0+re^{i\theta})|\,d\theta$$

$$\therefore \int_0^{2\pi}[\,|f(z_0)|-|f(z_0+re^{i\theta})|\,]d\theta=0.$$

Since the integrand in the above expression is continuous and non-negative,

we have $|f(z_0)| - |f(z_0 + re^{i\theta})| = 0.$

(i.e) $|f(z_0)| = |f(z_0 + re^{i\theta})|$ for all z in the circular disc $|z-z_0|<r.$

(i.e) $|f(z_0)| = |f(z)|$ for all z in the circular disc.

Therefore $f(z)$ is constant in a neighbourhood of z_0.

Since $f(z)$ is continuous, it follows that $f(z)$ is constant throughout D which is a contradiction.

Therefore the maximum of $|f(z)|$ is not attained at any of the interior points of D. Hence the theorem.

Solved Problems

Problem 8.2.7

Evaluate using Cauchy's integral formula

$$\frac{1}{2\pi i}\int_C \frac{z^2+5}{z-3}\,dz \qquad \text{where } C \text{ is } |z| = 4.$$

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Solution.

$f(z) = z^2 + 5$ is analytic inside and on $|z| = 4$ and $z = 3$ lies inside it.

Therefore by Cauchy's integral formula,

$$\frac{1}{2\pi i}\int_C \frac{z^2+5}{z-3}dz = f(3) = 3^2 + 5 = 14.$$

Problem 8.2.8

Evaluate $\int_C \frac{z}{z^2-1}dz$ where C is the positively oriented circle $|z| = 2$.

Solution.

$$\frac{1}{z^2-1} = \frac{1}{(z+1)(z-1)} = \frac{1}{2}\left(\frac{1}{z-1} - \frac{1}{z+1}\right).$$

$$\therefore \int_C \frac{z}{z^2-1}dz = \frac{1}{2}\int_C \frac{z}{z-1}dz - \frac{1}{2}\int_C \frac{zdz}{z+1}.$$

$f(z) = z$ is analytic and 1, -1 lie in the interior of C.

Therefore by Cauchy's integral formula,

$$\int_C \frac{z}{z\ \ -1}dz = 2\pi\ if(1) = 2\pi i.$$

Also $\int_C \frac{zdz}{z+1} = 2\pi\ i\ f(-1) = -2\pi i.$

$$\therefore \int_C \frac{zdz}{z^2-1} = \frac{1}{2}(2\pi\ i) - \frac{1}{2}(-2\pi\ i) = 2\pi\ i.$$

Problem 8.2.9

Evaluate $\int_C \frac{e^z}{z^2+4}dz$ where C is positively oriented circle $|z - i| = 2$

Solution.

$$\frac{1}{z^2+4} = \frac{1}{(z+2i)(z-2i)}$$

$$= \frac{1}{4i}\left(\frac{1}{z-2i} - \frac{1}{z+2i}\right) \quad \textit{(by partial fraction)}$$

Now, $2i$ les inside C and by Cauchy's integral formula, we have,

$$\int_C \frac{e^z}{z^2-2i}dz = 2\pi i e^{2i}.$$

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Also $-2i$ lies outside C and hence $\frac{e^z}{z+2i}$ is analytic inside and on C.

Hence by Cauchy's theorem, $\int_C \frac{e^z}{z+2i} dz = 0$

$$\therefore \int_C \frac{e^z}{z^2+4} dz = \frac{1}{4i}(2\pi i e^{2i} - 0) = \frac{\pi}{2} e^{2i}.$$

Problem 8.2.10

Evaluate $\int_C (\frac{\sin \pi z^2 + coz\pi z^2}{(z-1)(z-2)}) dz$ where C is the circle $|z| = 3$.

Solution.

By partial fractions,

$$\frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1}.$$

Let $f(z) = \sin \pi z^2 + \cos \pi z^2$. Then $f(z)$ is analytic inside and on C and the points 1 and 2 lie inside C. Hence by Cauchy's integral formula,

$$\int_C \frac{f(z)}{z-1} dz = 2\pi i f(1)$$

$$= 2\pi \, i(\sin \pi + \cos \pi)$$

$$= -2\pi i$$

Similarly,

$$\int_C \frac{f(z)}{z-2} dz = 2\pi i f(2)$$

$$= 2\pi i \,(\cos 4\pi + \sin 4\pi)$$

$$= 2\pi i$$

Hence $\int_C \frac{f(z)}{(z-1)(z-2)} dz = 2\pi i - (-2\pi i) = 4\pi i.$

Problem 8.2.11

Let C denote the boundary of the square whose sides lie along the lines $x = \pm 2$ and $y = \pm 2$ where C is described in the positive sense. Evaluate

$$(i) \int_C \frac{zdz}{2z+1} \quad (ii) \int_C \frac{\cos z}{z(z^2+8)}.$$

Solution.

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$$(i)\int_C \frac{zdz}{2z+1} = \frac{1}{2}\int_C \frac{zdz}{z+\frac{1}{2}}$$

$$= \frac{1}{2}(2\pi i)(-\frac{1}{2}) \quad (by\ Cauchy's\ integral\ formula)$$

$$= \frac{-\pi i}{2}.$$

$$(ii)\ let\ f(z) = \frac{\cos z}{z^2+8}.$$

The points where $f(z)$ is not analytic are $\pm i2\sqrt{2}$ and these points lie outside C. Hence $f(z)$ is analytic inside and on C.

By Cauchy's integral formula,

$$\int_C \frac{\cos z}{z(z^2+8)}dz = \int_C \frac{f(z)}{z}dz = 2\pi i f(0) = 2\pi i \left(\frac{1}{8}\right) = \frac{\pi i}{4}.$$

Problem 8.2.12

Evaluate $\frac{zdz}{(9-z^2)(z+i)}$ where C is the circle $|z| = 2$ taken in the positive sense.

Solution.

Let $f(z) = \frac{z}{9-z^2}.$

Clearly $f(z)$ is analytic within and on C.

By Cauchy's integral formula

$$\int_C \frac{zdz}{(9-z^2)(z+i)} = \int_C \frac{f(z)}{z+i}dz$$

$$= 2\pi i f(-i)$$

$$= 2\pi i\left(\frac{-i}{10}\right) = \frac{\pi}{5}.$$

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Exercises 8.2.13

1. Prove that $\int_C \frac{zdz}{z^2-1} = 2\pi i$ where C is the positively oriented circle $|z| = 2$.
2. Evaluate $\int_C \frac{dz}{z^2+4}$ where C is $|z-i| = 2$ in the positive sense.
3. Evaluate $\int_C \frac{e^z dz}{z^2+1}$ where C is the circle of radius 1 with centre at (i) $z = i$ and (ii) $z = -i$.
4. Evaluate $\int_C \frac{coz\pi z}{z^2-1}dz$ where C is a rectangle with vertices at (i) $2 \pm i$, $-2 \pm i$ and (ii) $-i$, $2-i$, $2+i$, i.
5. Show that $\frac{1}{2\pi i}\int_C \frac{e^{zt} dz}{z^2+1} = \sin t$ if $t > 0$ and C is the circle $|z| = 3$.
6. Evaluate $\int_C \frac{dz}{z^2(z-1)}$ where C is (i) $|z| = 3/4$ (ii) $|z| = 3/2$.

Answers: $2. \frac{\pi}{3}$ 3.(i)$\pi(\cos 1 + i\sin 1)$ (ii)$-\pi(\cos 1 - i\sin 1)$ 4. (i) 0 (ii) $-\pi i$
6. (i) $-2\pi i$ (ii) 0

8.3 HIGHER DERIVATIVES

In this section, we shall prove that an analytic function has derivatives of all orders. It follows, in particular, that *the derivative of an analytic function is again an analytic function.*

Consider a function $f(z)$ which is analytic in a region D. Let $z \in D$. Let C be any circle with centre z such that the circle and its interior is contained in D. By Cauchy's integral formula, we have,

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - z}d\zeta.$$

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We now proceed to prove that $f'(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z)^2} d\zeta.$

and in general $f^{(n)}(z) = \frac{n!}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta.$

Theorem 8.3.1

Let f be analytic inside and on a simple closed curve C.

Let z be any point inside C.

Then $f'(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z)^2} d\zeta.$

Proof.

By Cauchy's integral formula , we have $f(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z)} d\zeta.$

$$\therefore \frac{f(z+h) - f(z)}{h} = \frac{1}{h(2\pi i)}\int_C \left(\frac{f(\zeta)}{\zeta - z - h} - \frac{f(\zeta)}{\zeta - z}\right) d\zeta$$

$$= \frac{1}{h2\pi i}\int_C \left[\frac{hf(\zeta)}{(\zeta - z - h)(\zeta - z)}\right] d\zeta$$

$$= \frac{1}{2\pi i}\int_C \frac{f(\zeta)d\zeta}{(\zeta - z - h)(\zeta - z)} \quad ....(1)$$

Now,

$$\int_C \frac{f(\zeta)d\zeta}{(\zeta - z - h)(\zeta - z)} - \int_C \frac{f(\zeta)d\zeta}{(\zeta - z)^2}$$

$$= \int_C \left[\frac{f(\zeta)}{(\zeta - z - h)(\zeta - z)} - \frac{f(\zeta)}{(\zeta - z)^2}\right] d\zeta$$

$$= \int_C \frac{f(\zeta)}{(\zeta - z)}\left(\frac{1}{\zeta - z - h} - \frac{1}{\zeta - z}\right) d\zeta$$

$$= \int_C \frac{f(\zeta)}{(\zeta - z)}\left[\frac{h}{(\zeta - z - h)(\zeta - z)}\right] d\zeta$$

$$= h\int_C \frac{f(\zeta)d\zeta}{(\zeta - z - h)(\zeta - z)^2}$$

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$$\therefore \frac{1}{2\pi i}\int_C \frac{f(\zeta)d\zeta}{(\zeta - z - h)(\zeta - h)} - \frac{1}{2\pi i}\int_C \frac{f(\zeta)d\zeta}{(\zeta - z)^2} = \frac{h}{2\pi i}\int_C \frac{f(\zeta)d\zeta}{(\zeta - z - h)(\zeta - z)^2}$$

$$\therefore \frac{f(z+h) - f(z)}{h} - \frac{1}{2\pi i}\int_C \frac{f(\zeta)d\zeta}{(\zeta - z)^2}$$

$$= \frac{h}{2\pi i}\int \frac{f(\zeta)d\zeta}{(\zeta - z - h)(\zeta - z)^2} \quad (\text{using}(1)) \quad ....(2)$$

Now, let M denote the maximum value of $|f(\zeta)|$ on C. Let L be the length of C and d be the shortest distance from z to any point on the curve C.
∴

For any point ζ on C we have,

$$|\zeta - z| \geq d \text{ and } |\zeta - z - h| \geq |\zeta - z| - |h| \geq d - |h|$$

Hence

$$\left|\frac{f(\zeta)}{(\zeta - z)^2(\zeta - z - h)}\right| \leq \frac{M}{d^2(d - |h|)}$$

From (2) we get

$$\left|\frac{f(z+h) - f(z)}{h} - \frac{1}{2\pi i}\int_C \frac{f(\zeta)d\zeta}{(\zeta - z)^2}\right| \leq \frac{|h|}{2\pi}\left(\frac{M}{d^2(d - |h|}\right)$$

$$\therefore \lim_{h\to 0}\left(\frac{f(z+h) - f(z)}{h} - \frac{1}{2\pi i}\int_C \frac{f(\zeta)d\zeta}{(\zeta - z)^2}\right) = 0$$

$$\therefore \lim_{h\to 0}\frac{f(z+h) - f(z)}{h} = \frac{1}{2\pi i}\int_C \frac{f(\zeta)d\zeta}{(\zeta - z)^2}$$

$$\therefore f'(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)d\zeta}{(\zeta - z)^2}$$

Remark 8.3.2

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By using induction on n, we can prove that for any positive integer n, we have,

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta.$$

Note 8.3.3

Thus an analytic function has derivatives of all orders and the derivate of an analytic function is again analytic.

Example 8.3.3

$$\int_C \frac{e^z}{z^n} dz = \frac{2\pi i}{(n-1)!} \text{ where } C \text{ is the circle } |z| = 1.$$

Solution.

Let $f(z) = e^z$. Clearly $f(z)$ is analytic and $f^{(n)}(z) = e^z$ for all n.

By the formula for higherderivatives,

$$\int_C \frac{e^z}{z^n} dz = \int_C \frac{e^z}{(z-0)^n} dz = \frac{2\pi i}{(n-1)!} e^0 = \frac{2\pi i}{(n-1)!}$$

Example 8.3.4

$$\int_C \frac{\sin^2 z}{(z - \pi/6)^3} dz = \pi i \text{ where } C \text{ is the circle } |z| = 1.$$

Solution.

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Let $f(z) = sin^2 z$. Then $f'(z) = 2 \sin z \cos z = \sin 2z$.

$f''(z) = 2 \cos 2z$. Also $\pi/6$ lies inside C.

$$\therefore \int \frac{\sin^2 z}{(z-\pi/6)^3} dz = \frac{2\pi i}{2!} f''(\pi/6)$$

$$= \pi i (2 \cos \pi/3)$$

$$= \pi i.$$

Theorem 8.3.5 (Cauchy's Inequality)

Let $f(z)$ be analytic inside and on the circle C with centre z_0 and radius r. Let M denote the maximum of $|f(z)|$ on C.

Then $|f^n(z_0)| \le \dfrac{n!M}{r^n}$.

Proof.

We have

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z)dz}{(z-z_0)^{n+1}}$$

$$\therefore \left|f^{(n)}(z_0)\right| \le \frac{n!}{2\pi} \left(\frac{M}{r^{n+1}} \right) (2\pi r) = \frac{n!M}{r^n}$$

$$Hence \left|f^{(n)}(z_0)\right| \le \frac{n!M}{r^n}$$

Theorem 8.3.6 (Liouville's Theorem)

A bounded entire function in the complex plane is constant.

Proof.

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Let $f(z)$ be a bounded entire function.

Since $f(z)$ is bounded, there exists a real number M such that $|f(z)| \leq M$ for all z.

Let z_0 be any complex number and $r > 0$ be any real number.

By Cauchy's inequality, we have

$$|f'(z_0)| \leq \frac{M}{r}.$$

Taking the limit as $r \to \infty$ we get, $f'(z_0) = 0$.

Since z_0 is arbitrary $f'(z) = 0$ for all z in the complex plane.

Therefore $f(z)$ is a constant function.

Theorem 8.3.7 (Fundamental Theorem of Algebra)

Every polynomial of degree ≥ 1 has atleast one zero (root) in **C**.

Proof.

Let $f(z)$ be a polynomial of degree ≥ 1.

Suppose $f(z)$ has no zero in **C**. Then $f(z) \neq 0$ for all z.

Further $f(z)$ is an entire function in the complex plane.

$\therefore \frac{1}{f(z)}$ is also an entire function. Also as $z \to \infty$, $f(z) \to \infty$.

$\therefore \frac{1}{f(z)} \to 0$ as $z \to \infty$.

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$\therefore \frac{1}{f(z)}$ is a bounded function.

Hence by Liouville's theorem, $\frac{1}{f(z)}$ is constant function.

$\therefore f(z)$ is a constant function and hence it is a polynomial of degree zero which is a contradiction.
Hence $f(z)$ has atleast one root in **C**.
Hence the theorem.

Theorem 8.3.8 (Morera's Theorem)

If $f(z)$ is continuous in a simply connected domain D and if $\int_C f(z)dz = 0$ for every simple closed curve C lying in D, then $f(z)$ is analytic in D.

(This theorem is the converse of Cauchy's theorem)

Proof.

By Corollary 8.1.5 , there exists an analytic function $F(z)$ such that $F'(z) = f(z)$ in D.
Also we know the derivative of an analytic function is an analytic function.
Hence $F'(z)$ is analytic in D.

$\therefore f(z)$ is analytic in D.

Solved Problems

Problem 8.3.9

Evaluate $\int_C \frac{\sin z}{(z-\pi/2)^2}dz$ where C is the circle $|z| = 2$.

Solution.

Let $f(z) = \sin z$. Hence $f'(z) = \cos z$. Also $\pi/2$ lies inside $|z| = 2$.

Hence $\int_C \frac{\sin z}{(z-\pi/2)^2}dz = 2\pi i f'(\pi/2)$

$= 2\pi i(coz\ \pi/2)$

$= 0.$

Problem 8.3.10

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Evaluate $\int_C \frac{z^3}{(2z+i)^3}dz$ where C is the unit circle.

Solution.

$$\int_C \frac{z^3}{(2z+i)^3}dz = \frac{1}{8}\int_C \frac{z^3}{(z+i/2)^3}dz$$

Let $f(z) = z^3$. Then $f'(z) = 3z^2$ and $f''(z) = 6z$.

Also $-\frac{i}{2}$ lies inside C.

Hence $\int_C \frac{z^3}{(2z+i)^3}dz = \frac{1}{8}\left(\frac{2\pi i}{2!}\right)f''\left(-\frac{i}{2}\right)$

$$= \frac{2\pi i}{16}(-3i)$$

$$= \frac{3\pi}{8}$$

Problem 8.3.11

Evaluate $\int_C \frac{(e^z + z\sinh z)}{(z-\pi i)^2}dz$ where C is the circle $|z| = 4$.

Solution. Let $f(z) = e^z + z \sinh z$.

Therefore $f'(z) = e^z + z \cosh z + \sinh z$.

Also πi lies inside C.

Hence $\int_C \frac{f(z)}{(z-\pi i)^2}dz = 2\pi i f'(\pi i)$.

$$= 2\pi i[e^{\pi i} + \pi i \cosh \pi i + \sinh \pi i]$$

$$= 2\pi i(-1 - \pi i)$$

$$= -2\pi i(1 + \pi i).$$

Problem 8.3.12

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Show that when f is analytic within and on a simple closed curve and z_0

is not on C,

then $\int_C \frac{f'(z)dz}{(z-z_0)} = \int_C \frac{f(z)dz}{(z-z_0)^2}$

Case i. Suppose z_0 is in the exterior of C. Then both $\frac{f(z)}{(z-z_0)^2}$ and $\frac{f'(z)}{(z-z_0)}$ are analytic inside and on C.

Therefore by Cauchy's theorem, $\int_C \frac{f'(z)dz}{(z-z_0)} = \int_C \frac{f(z)dz}{(z-z_0)^2} = 0$

Case ii. z_0 lies in the interior.

Then by Cauchy's integral formula, $\int_C \frac{f'(z)dz}{(z-z_0)} = 2\pi i f'(z_0)$.

Also by the formula for higher derivatives, $\int_C \frac{f(z)dz}{(z-z_0)^2} = 2\pi i f'(z_0)$.

Hence $\int_C \frac{f'(z)dz}{(z-z_0)} = \int_C \frac{f(z)dz}{(z-z_0)^2}$.

Problem 8.3.13

Let the function $f(z) = u(x,y) + iv(x,y)$ be continuous in a closed bounded region D and let it be analytic and not constant in the interior of D. Show that the function $u(x,y)$ reaches its maximum value on the boundary of D and never in the interior of D.

Solution.

Consider the function $e^{f(z)}$. Since $f(z)$ is continuous in a closed bounded region D and non constant in the interior of D, $e^{f(z)}$ is also continuous in the closed bounded region D and analytic and non constant in the interior of D.

Now, the maximum value of $|e^{f(z)}|$ is attained only at a boundary point of D. (by maximum modulus theorem).

Therefore $|e^{f(z)}| = e^{u(x,y)}$

Therefore, maximum value $e^{u(x,y)}$ is attained only at a boundary point of D.

Therefore, maximum value of $u(x,y)$ is attained only at a boundary point of D.

Problem 8.3.14

Evaluate $\int_C \frac{\sin 2z\, dz}{(z-\pi i/4)^4}$ where C is $|z| = 1$.

Solution.

Space for Hints

Let $f(z) = \sin 2z$. Since $f(z)$ is analytic and $\pi i/4$ lies inside C, we have,

$$\int_C \frac{\sin 2z\, dz}{(z-\pi i/4)^4} = \frac{2\pi i}{3!} f'''\left(\frac{\pi i}{4}\right)$$

Now $f'(z) = 2\cos 2z$, $f''(z) = -4\sin 2z$, $f'''(z) = -8\cos 2z$.

Hence $f'''(\pi i/4) = -8\cos(\pi i/2)$

$= -8\cosh(\pi/2)$.

$$\therefore \int_C \frac{\sin 2z\, dz}{(z-\pi i)^4} = -\frac{8\pi i}{3}\cosh\left(\frac{\pi}{2}\right)$$

Problem 8.3.15

Evaluate $\int_C \frac{e^{2z}}{(z+1)^4}dz$ where C is the circle $|z| = 2$.

Solution.

Let $f(z) = e^{2z}$. Clearly $f(z)$ is analytic and

$f'(z) = 2e^{2z}$; $f''(z) = 4e^{2z}$; $f'''(z) = 8e^{2z}$

By the formula for higher derivatives,

$$\int_C \frac{e^{2z}}{(z+1)^4}dz = \left(\frac{2\pi i}{3!}\right)f'''(-1)$$

$$= \left(\frac{2\pi i}{6}\right)(8e^{-2})$$

$$= \frac{i8\pi e^{-2}}{3}.$$

Problem 8.3.16

Evaluate $\int_C \frac{e^z}{(z+2)(z+1)^2}dz$ where C is $|z| = 3$.

Solution.

$$\frac{1}{(z+2)(z+1)^2} = \frac{(z+2)-(z+1)}{(z+2)(z+1)^2}$$

$$= \frac{1}{(z+1)^2} - \frac{1}{(z+2)(z+1)}$$

$$= \frac{1}{(z+1)^2} - \frac{1}{z+1} + \frac{1}{z+2}.$$

$$\int_C \frac{e^z}{(z+2)(z+1)^2}dz = \int_C \frac{e^z}{(z+2)}dz - \int_C \frac{e^z}{(z+1)}dz + \int_C \frac{e^z}{(z+1)^2}dz \quad \text{--------------------------------(1)}$$

We note that $z = -2, -1$ lie in the interior of C.

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Let $f(z) = e^z$. It is analytic in C. Also $f'(z) = e^z$.

By Cauchy's integral formula,

$$\int_C \frac{e^z}{(z+2)} dz = 2\pi i f(-2) = 2\pi i e^{-2}.$$

$$\int_C \frac{e^z}{(z+1)} dz = 2\pi i f(-1) = 2\pi i e^{-1}.$$

$$\int_C \frac{e^z}{(z+1)^2} dz = \left(\frac{2\pi i}{1!}\right) f'(-1) = 2\pi i e^{-1}.$$

Therefore from (1), $\int_C \frac{e^z}{(z+2)(z+1)^2} dz = 2\pi i[e^{-2} - e^{-1} + e^{-1}]$

$= 2\pi i e^{-2}$.

Exercises 8.3.17

Evaluate the following.

1. $\int_C \frac{(z+2)dz}{z^2}$ where C is $|z| =$

2. $\int_C \frac{e^{2z}}{(z-1)^4} dz$ where C is $|z| = 3/2$.

3. $\int_C \frac{\cos z \, dz}{(z-\pi/2)^2}$ where C is $|z| = 2$.

4. $\int_C \frac{e^{az}}{z^{n+1}} dz$ where C is $|z| = 1/2$.

UNIT 9
SERIES EXPANSIONS

Space for Hints

INTRODUCTION

In this chapter, we consider the problem of representing a given function as a power series. We prove that if a function is analytic at a point z_0, then it can be expanded as a power series called ***Taylor's series*** consisting of non-negative powers of $z - z_0$ and the expansion is valid in some neighbourhood of z_0. We also prove that a function $f(z)$ which is analytic in an annular region $a < |z - z_0| < b$ can be expanded as a series called ***Laurent's series*** consisting of positive and negative powers of $z - z_0$. We also introduce the concept of ***singular points*** of a function and classify the singular points and discuss the behavior of the function in the neighbourhood of a singularity.

9.1 TAYLOR'S SERIES

Theorem 9.1.1(Taylor's Theorem)

Let $f(z)$ be analytic in a region D containing z_0. Then $f(z)$ can be represented as a power series in $z - z_0$ given by

$$f(z) = f(z_0) + \frac{f'(z_0)}{1!}(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \dots\dots\dots + \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n + \dots\dots$$

The expansion is valid in the largest open disc with centre z_0 contained in D.

Proof.

Let $r > 0$ be such that the disc $|z - z_0| < r$ is contained in D.

Let $0 < r_1 < r$. Let C_1 be the circle $|z - z_0| = r_1$.

By Cauchy's integral formula, we have,

$$f(z) = \frac{1}{2\pi i}\int_{C_1} \frac{f(\zeta)}{(\zeta - z)} d\zeta \qquad \dots\dots\dots\dots(1)$$

Also by theorem on higher derivatives, we have,

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_{C_1} \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta \qquad \dots\dots\dots\dots(2)$$

Now,

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$$\frac{1}{\zeta - z} = \frac{1}{(\zeta - z_0) - (z - z_0)}$$

$$= \frac{1}{(\zeta - z_0)\left[1 - \frac{z - z_0}{\zeta - z_0}\right]}$$

$$= \frac{1}{\zeta - z_0}\left[1 + \left(\frac{z - z_0}{\zeta - z_0}\right) + \left(\frac{z - z_0}{\zeta - z}\right)^2 + \ldots\ldots + \left(\frac{z - z_0}{\zeta - z_0}\right)^{n-1} + \frac{\left(\frac{z - z_0}{\zeta - z_0}\right)^n}{1 - \left(\frac{z - z_0}{\zeta - z_0}\right)}\right]$$

(using the identify $\frac{1}{1-\alpha} = 1 + \alpha + \alpha^2 + \ldots\ldots + \alpha^{n-1} + \frac{\alpha^n}{1-\alpha}$)

$$= \frac{1}{\zeta - z_0} + \frac{z - z_0}{(\zeta - z_0)^2} + \frac{(z - z_0)^2}{(\zeta - z_0)^3} + \ldots\ldots + \frac{(z - z_0)^{n-1}}{(\zeta - z_0)^n} + \frac{(z - z_0)^n}{(\zeta - z_0)^n(\zeta - z)}.$$

Now, multiplying throughout by $\frac{f(\zeta)}{2\pi i}$, integrating over C_1 and using (1)

and (2),

we get

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \ldots\ldots + \frac{f^{(n-1)}(z_0)}{(n-1)!}(z - z_0)^{n-1} + R_n \ldots\ldots(3)$$

Where $R_n = \frac{(z - z_0)^n}{2\pi i}\int_{C_1}\frac{f(\zeta)d\zeta}{(\zeta - z)(\zeta - z_0)^n}$

Here ζ lies on C_1 and z lies in the interior of C_1 so that $|\zeta - z_0| = r_1$ and $|z - z_0| < r_1$.

$\therefore |\zeta - z| = |(\zeta - z_0) - (z - z_0)| \geq |\zeta - z_0| - |z - z_0| = r_1 - |z - z_0|$.

$$\therefore \frac{1}{|\zeta - z|} \leq \frac{1}{r_1 - |z - z_0|}$$

Let M denote the maximum value of $|f(z)|$ on C_1.

Then by Theorem: $\left|\int_C f(z)dz\right| \leq Ml \text{ where } M = \max\{|f(z)| : z \in C\}$

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$$|R_n| \le \frac{|z-z_0|^n}{2\pi}\frac{M(2\pi r_1)}{(r_1-|z-z_0|)r_1^{\,n}}$$

$$= \frac{M|z-z_0|}{(r_1-|z-z_0|)}\left(\frac{|z-z_0|}{r_1}\right)^{n-1}$$

Also $\left|\frac{z-z_0}{r_1}\right| < 1$. Hence $\lim_{n\to\infty} R_n = 0$.

$\therefore$ Taking limit as n $\to \infty$ in (3),we get

$$f(z)=f(z_0)+\frac{f'(z_0)}{1!}(z-z_0)+\frac{f''(z_0)}{2!}(z-z_0)^2+........+\frac{f^{(n)}(z_0)}{n!}(z-z_0)^n+..$$

Note 9.1.2

The above series is called the **Taylor series** of *f(z)* about the point z_0. Thus if *f(z)* is analytic at a point z_0, then *f(z)* can be represented as a Taylor's series about z_0, which is a series in non negative powers of $z - z_0$. The expansion is valid in some neighbourhood of z_0.

Note 9.1.2

The Taylor series expansion of *f(z)* about the point zero is called the ***Maclaurin's series***. Thus the Maclaurin's series of *f(z)* is given by

$$f(z)=f(0)+\frac{z}{1!}f'(0)+\frac{z^2}{2!}f''(0)+........+\frac{z^n}{n!}f^{(n)}(0)+.......$$

Example 9.1.3

The Taylor's series for $f(z) = \frac{1}{z}$ about $z = 1$ is given by

$$\frac{1}{z}=f(1)+\frac{f'(1)}{1!}(z-1)+\frac{f''(1)}{2!}(z-1)^2+\frac{f'''(1)}{3!}(z-1)^3+.......$$

Now,

$$f(z)=\frac{1}{z}\Rightarrow f(1)=1$$

$$f'(z)=-\frac{1}{z^2}\Rightarrow f'(1)=-1$$

$$f''(z)=\frac{2}{z^3}\Rightarrow f''(1)=2$$

$$f'''(z)=-\frac{6}{z^4}\Rightarrow f'''(1)=-6$$

....

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Hence the Taylor's series expansion for $\frac{1}{z}$ about 1 is

$$\frac{1}{z} = 1 - (z-1) + (z-1)^2 - (z-1)^3 + \dots\dots\dots$$

This expansion is valid in the disc $|z-1| < 1$.

Similarly Taylor's series for $f(z) = \frac{1}{z}$ about $z = i$ is given by

$$\frac{1}{z} = \frac{1}{i} - \frac{z-i}{i^2} + \frac{(z-i)^2}{i^3} - \frac{(z-i)^3}{i^4} + \dots\dots$$

Example 9.1.4

Let $f(z) = e^z$.

Then $f^{(n)} = e^z$ for all n and hence $f^{(n)}(0) = 1$.

Hence the Maclaurin's series for e^z is given by

$$e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \dots\dots + \frac{z^n}{n!} + \dots\dots$$

and the expansion is valid in the entire complex plane.

Maclaurin's series expansion of some of the standard functions are given below.

1. $e^{-z} = 1 - \frac{z}{1!} + \frac{z^2}{2!} - \dots\dots + (-1)^n \frac{z^n}{n!} + \dots\dots (|z| < \infty)$

2. $\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots\dots + (-1)^{n-1} \frac{z^{2n-1}}{(2n-1)!} + \dots\dots (|z| < \infty)$

3. $\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \dots\dots + (-1)^{n-1} \frac{z^{2n-2}}{(2n-2)!} + \dots\dots (|z| < \infty)$

4. $\sinh z = \frac{1}{1!} + \frac{z^3}{3!} + \frac{z^5}{5!} - \dots\dots + \frac{z^{2n-1}}{(2n-1)!} + \dots\dots (|z| < \infty)$

5. $\cosh z = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} - \dots\dots + \frac{z^{2n}}{(2n)!} + \dots\dots (|z| < \infty)$

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6. $\dfrac{1}{1+z}=1-z+z^2-z^3+........+(-1)^n z^n+.......(|z|<1)$

7. $\dfrac{1}{1-z}=1+z+z^2+z^3+........+z^n+.......(|z|<1)$

8. $\log(1+z)=z-\dfrac{z^2}{2}+\dfrac{z^3}{3}-........+(-1)^{n-1}\dfrac{z^n}{n}+.......(|z|<1)$

9. $\log(1-z)=-z-\dfrac{z^2}{2}-\dfrac{z^3}{3}-........-\dfrac{z^n}{n}-.......(|z|<1)$

Solved Problems

Problem 9.1.5

Expand *cos z* into a Taylor's series about the point $z = \pi/2$ and determine the region of convergence.

Solution.

Let $f(z) = \cos z$.

The Taylor's series for $f(z)$ about $z = \pi/2$ is

$$f(z)=f(\pi/2)+\frac{f'(\pi/2)}{1!}(z-\pi/2)+\frac{f''(\pi/2)}{2!}(z-\pi/2)^2+\frac{f'''(\pi/2)}{3!}(z-\pi/2)^3+..$$

Now $f(z) = \cos z$. *Hence* $f(\pi/2) = 0$.

$f'(z) = -\sin z$. *Hence* $f'(\pi/2) = -1$.

$f''(z) = -\cos z$. *Hence* $f''(\pi/2) = 0$.

$f'''(z) = \sin z$. *Hence* $f'''(\pi/2) = 1$.

....

The Taylor's series for *cos z* about $z = \pi/2$ is

$$\cos z=-\frac{(z-\pi/2)}{1!}+\frac{(z-\pi/2)^3}{3!}-\frac{(z-\pi/2)^5}{5!}+.......$$

The expansion is valid throughout the complex plane.

Problem 9.1.6

Expand $f(z) = \sin z$ in a Taylor's series about $\pi/4$ and determine the region of convergence of this series.

Solution.

The Taylor's series for $f(z)$ about $z = \pi/4$ is

Space for Hints

$$f(z)=f(\pi/4)+\frac{f'(\pi/4)}{1!}(z-\pi/4)+\frac{f''(\pi/4)}{2!}(z-\pi/4)^2+\frac{f'''(\pi/4)}{3!}(z-\pi/4)^3+.......$$

Here $f(z) = \sin z$. Hence $f(\pi/4) = 1/\sqrt{2}$.

$f'(z) = \cos z$. Hence $f'(\pi/4) = 1/\sqrt{2}$.

$f''(z) = -\sin z$. Hence $f''(\pi/4) = -1/\sqrt{2}$.

$f'''(z) = -\cos z$. Hence $f'''(\pi/4) = -1/\sqrt{2}$.

The Taylor's series for $\sin z$ about $z = \pi/4$ is

$$\sin z=\frac{1}{\sqrt{2}}+\frac{(z-\pi/4)}{1!}\left(\frac{1}{\sqrt{2}}\right)-\frac{(z-\pi/4)^2}{2!}\left(\frac{1}{\sqrt{2}}\right)-\frac{(z-\pi/4)^3}{3!}\left(\frac{1}{\sqrt{2}}\right)+.......$$

$$=\frac{1}{\sqrt{2}}\left(1+\frac{(z-\pi/4)}{1!}-\frac{(z-\pi/4)^2}{2!}-\frac{(z-\pi/4)^3}{3!}+.......\right)$$

The expansion is valid in the entire complex plane.

Problem 9.1.7

Expand $f(z)=\frac{z-1}{z+1}$ as a Taylor's series

(a) about the point $z = 0$.

(b) about the point $z = 1$. Determine the region of convergence in each case.

Solution.

a. $f(z)=\frac{z-1}{z+1}$

$= (z-1)(z+1)^{-1}$

$= (z-1)(1-z+z^2-z^3+........)$ if $|z| < 1$.

$= (z-z^2+z^3-....)-(1-z+z^2-z^3-.....)$

$= -1+2z-2z^2+2z^3+........$

The region convergence is $|z|<1$.

b. $f(z)=\frac{z-1}{z+1}$

$$= \frac{z-1}{2+z-1}$$

$$= \frac{z-1}{2\left(1+\frac{z-1}{2}\right)}$$

$$= \frac{z-1}{2}\left(1+\frac{z-1}{2}\right)^{-1}$$

$$= \frac{z-1}{2}\left[1-\frac{z-1}{2}+\left(\frac{z-1}{2}\right)^2-\left(\frac{z-1}{2}\right)^3+\ldots\ldots\right] if \left|\frac{z-1}{2}\right|<1$$

$$= \frac{z-1}{2}-\frac{(z-1)^2}{2^2}+\frac{(z-1)^3}{2^3}-\ldots\ldots\ldots$$

The region of convergence is given by $\left|\frac{z-1}{2}\right| < 1$ which is same as the circular disc $|z-1| < 2$.

Problem 9.1.8

Show that

i. $$\frac{1}{z^2} = 1+\sum_{n=1}^{\infty}(n+1)(z+1)^n \quad when|\, z+1\,|<1$$

ii. $$\frac{1}{z^2} = \frac{1}{4}+\frac{1}{4}\sum_{n=1}^{\infty}(-1)^n(n+1)\left(\frac{z-2}{2}\right)^n \quad when|\, z-2\,|<2$$

Solution.

i. $$\frac{1}{z^2} = \frac{1}{[1-(z+1)]^2}$$

$$= [1-(z+1)]^{-2}$$

$$= 1 + 2(z+1) + 3(z+1)^2+4(z+1)^3+\ldots\ldots \; if\, |z+1| < 1.$$

$$= 1+\sum_{n=1}^{\infty}(n+1)(z+1)^n \quad when|\, z+1\,|<1$$

ii. $$\frac{1}{z^2} = \frac{1}{(z-2+2)^2}$$

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Space for Hints

$$=\frac{1}{\left[2\left(1+\frac{z-2}{2}\right)\right]^2}$$

$$=\frac{1}{4}\left[1-2\left(\frac{z-2}{2}\right)+3\left(\frac{z-2}{2}\right)^2-.......\right] if \left|\frac{z-2}{2}\right|<1$$

$$=\frac{1}{4}+\frac{1}{4}\sum_{n=1}^{\infty}(-1)^n(n+1)\left(\frac{z-2}{2}\right)^n$$

Here the region of convergence is $\left|\frac{z-2}{2}\right|<1$ which is the same as the circular disc $|z-2|<2$.

Problem 9.1.9

Expand ze^{2z} in a Taylor's series about $z = -1$ and determine the region of convergence.

Solution.

Let $f(z) = ze^{2z}$.

$= ze^{2(z+1)}e^{-2}$

$$=\frac{1}{e^2}\left[(z+1)e^{2(z+1)}-e^{2(z+1)}\right]$$

$$=\frac{1}{e^2}\left[(z+1)\left\{1+\frac{2(z+1)}{1!}+\frac{4(z+1)^2}{2!}+\right\}-\left\{1+\frac{2(z+1)}{1!}+\frac{4(z+1)^2}{2!}\right\}\right]$$

$$=\frac{1}{e^2}\left[\left\{(z+1)+\frac{2(z+1)^2}{1!}+\frac{2^2(z+1)^3}{2!}+....\right\}-\left\{1+\frac{2(z+1)}{1!}+\frac{2^2(z+1)^2}{2!}+.....\right\}\right]$$

$$=\frac{1}{e^2}\left[-1+\left(1-\frac{2}{1!}\right)(z+1)+\left(\frac{2}{1!}-\frac{2^2}{2!}\right)(z+1)^2+\left(\frac{2^2}{2!}-\frac{2^3}{3!}\right)(z+1)^3+.......\right]$$

The expansion is valid throughout the complex plane.

Problem 9.1.10

Find the Taylor's series to represent $\frac{z^2-1}{(z+2)(z+3)}$ in $|z|<2$.

Solution. By partial fractions

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$$\frac{z^2-1}{(z+2)(z+3)} = 1+\frac{3}{z+2}-\frac{8}{z+3}(verify)$$

$$= 1+\frac{3}{2\left(1+\frac{z}{2}\right)}-\frac{8}{3\left(1+\frac{z}{3}\right)}$$

$$= 1+\frac{3}{2}\left(1+\frac{z}{2}\right)^{-1}-\frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$$

$$= 1+\frac{3}{2}\left(1-\frac{z}{2}+\frac{z^2}{2^2}-....\right)-\frac{8}{3}\left(1-\frac{z}{3}+\frac{z^2}{3^2}-....\right)$$

$$= \left(1+\frac{3}{2}-\frac{8}{3}\right)+\left(-\frac{3}{2^2}+\frac{8}{3^2}\right)z+\left(\frac{3}{2.2^2}-\frac{8}{3.3^2}\right)z^2+......$$

$$= -\frac{1}{6}+\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{8}{3^{n+1}}-\frac{3}{2^{n+1}}\right)z^n$$

and the expansion is valid in $|z| < 2$.

Exercises 9.1.11

1.Expand $1/z$ about $z = -1$ and $z = 2$ as Taylor's series, stating the region of convergence.

2.Find the Taylor's series for ze^z about $z = 1$.

Answers: 1. $1/z = -1-(z-1)-(z-1)^2+....:|z+1|$

2. $e\left[1+\frac{2(z-1)}{1!}+\frac{3(z-1)^2}{2!}+\frac{4(z-1)^3}{3!}+...\right]$

9.2 LAURENT'S SERIES

A series of the form $\sum_{n=1}^{\infty}\frac{b_n}{z_n}$ ....(1)

can be considered as an ordinary power series in the variable $1/z$. Hence if the radius of convergence of the power series $\sum_{n=1}^{\infty} b_n z^n$ is r and $r < \infty$, then the series $\sum_{n=1}^{\infty}\frac{b_n}{z_n}$ converges in the region $|z| > r$. The convergence is uniform in every region $|z| \geq \rho > r$ and the series represents an analytic function in $|z| > r$.

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If the series (1) is combined with the usual power series, we get a more general series of the form $\sum_{-\infty}^{\infty} a_n z^n$ . ...(2)

This series is said to ***converge*** at a point if the part of the series consisting of the negative powers of z and the part of the series consisting of non-negative powers of z and separately convergent. We know that the series consisting of non-negative powers of z converges in a disc $|z| < r_2$ and the series consisting of negative powers of z converges in a region $|z| > r_1$.

Therefore if $r_1 < r_2$ the series represented by (2) converges in the region $r_1 < |z| < r_2$ and in this ***annulus region*** it represents an analytic function.

We shall now prove that the converse situation is also true.

(i.e) any function which is analytic in a region containing the annulus $r_1<|z-z_0|<r_2$ can be represented in a series of the form $\sum_{-\infty}^{\infty} a_n (z-z_0)^n$.

Theorem 9.2.1 (Laurent's Theorem)

Let C_1 and C_2 denote respectively the concentric circles $|z - z_0| = r_1$ and $|z - z_0| = r_2$ with $r_1 < r_2$. Let $f(z)$ be analytic in a region containing the circular annulus $r_1<|z - z_0|<r_2$. Then $f(z)$ can be represented as a convergent series of positive and negative powers of $z - z_0$ given by

$$f(z) = \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n} + \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

Where $b_n = \frac{1}{2\pi i}\int_{C_1} \frac{f(\zeta)d\zeta}{(\zeta - z_0)^{-n+1}}$ and $a_n = \frac{1}{2\pi i}\int_{C_2} \frac{f(\zeta)d\zeta}{(\zeta - z_0)^{n+1}}$

Proof.

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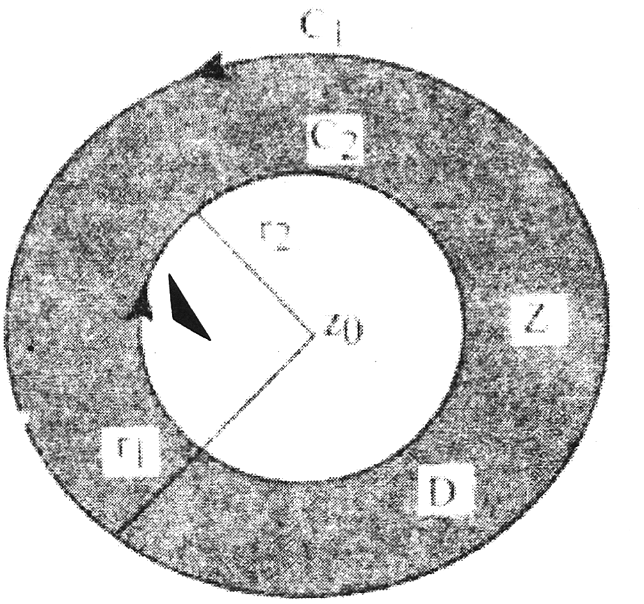

Let z be any point in the circular annulus $r_1<|z-z_0|<r_2$.
Then by Theorem 8.2.2,

we have, $f(z) = \dfrac{1}{2\pi i}\displaystyle\int_{C_2}\frac{f(\zeta)d\zeta}{\zeta - z} - \frac{1}{2\pi i}\int_{C_1}\frac{f(\zeta)d\zeta}{\zeta - z}$.

$$\therefore f(z) = \frac{1}{2\pi i}\int_{C_2}\frac{f(\zeta)d\zeta}{\zeta - z} + \frac{1}{2\pi i}\int_{C_1}\frac{f(\zeta)d\zeta}{z-\zeta} \qquad \text{.......(1)}$$

As in the proof of Taylor's theorem, we have

$$\frac{1}{2\pi i}\int_{C_2}\frac{f(\zeta)d\zeta}{\zeta - z} = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \text{.....} + a_{n-1}(z-z_0)^{n-1} + R_n(z)$$

......(2)

Where $a_n = \dfrac{1}{2\pi i}\displaystyle\int_{C_2}\frac{f(\zeta)d\zeta}{(\zeta - z_0)^{n+1}}$ and $R_n(z) = \dfrac{(z-z_0)^n}{2\pi i}\displaystyle\int_{C_2}\frac{f(\zeta)d\zeta}{(\zeta-z_0)^n(\zeta - z)}$

$$\frac{1}{z-\zeta} = \frac{1}{z-z_0+z_0-\zeta}$$

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$$= \frac{1}{(z-z_0)-(\zeta-z_0)}$$

$$= \frac{1}{(z-z_0)\left[1-\dfrac{\zeta-z_0}{z-z_0}\right]}$$

$$= \frac{1}{z-z_0}\left[1+\left(\frac{\zeta-z_0}{z-z_0}\right)+\left(\frac{\zeta-z_0}{z-z_0}\right)^2+\ldots\ldots+\left(\frac{\zeta-z_0}{z-z_0}\right)^{n-1}+\frac{\left(\dfrac{\zeta-z_0}{z-z_0}\right)}{1-\left(\dfrac{\zeta-z_0}{z-z_0}\right)}\right]$$

Multiplying by $\dfrac{f(\zeta)}{2\pi i}$ and integrating C_1 we get,

$$\int_{C_1}\frac{f(\zeta)d\zeta}{z-\zeta} = \frac{b_1}{z-z_0}+\frac{b_2}{(z-z_0)^2}+\ldots\ldots+\frac{b_{n-1}}{(z-z_0)^{n-1}}+S_n(z) \quad \ldots\ldots(3)$$

Where $b_n = \dfrac{1}{2\pi i}\displaystyle\int_{C_1}\frac{f(\zeta)d\zeta}{(\zeta-z_0)^{-n+1}}$; $S_n = \dfrac{1}{2\pi i(z-z_0)^n}\displaystyle\int_{C_1}\frac{f(\zeta)(\zeta-z_0)^n d\zeta}{z-\zeta}$

From (1), (2) and (3) we get

$$f(z) = a_0 + a_1(z-z_0) + \ldots\ldots + a_{n-1}(z-z_0)^{n-1} + \frac{b_1}{z-z_0}+\frac{b_2}{(z-z_0)^2}+\ldots\ldots+\frac{b_{n-1}}{(z-z_0)^{n-1}}+R_n(z)+S_n(z) \quad \ldots\ldots(4)$$

The required result follows if we can prove that $R_n \to 0$ and $S_n \to 0$ as $n \to \infty$.

Now, if $\zeta \in C_1$, then $|\zeta - z_0| = r_1$ and

$|z-\zeta| = |(z-z_0)-(\zeta-z_0)| \geq |z-z_0| - r_1$.

If $\zeta \in C_2$, then $|\zeta - z_0| = r_2$ and

$|\zeta - z| = |(\zeta-z_0)-(z-z_0)| \geq r_2 - |z-z_0|$

Now let M denote the maximum value of $|f(z)|$ in $C_1 \cup C_2$.

Then

by theorem $\left|\displaystyle\int_C f(z)dz\right| \leq Ml$ *where* $M = \max\{|f(z)| : z \in C\}$, *we have,*

$$|R_n| \leq \frac{|z-z_0|^n}{2\pi}\frac{M(2\pi r_2)}{r_2^n(r_2-|z-z_0|)}$$

$$\leq \frac{M|z-z_0|}{(r_2-|z-z_0|)}\left(\frac{|z-z_0|}{r_2}\right)^{n-1}$$

Since $\dfrac{|z-z_0|}{r_2} < 1$, $R_n \to 0$ as $n \to \infty$.

Space for Hints

Also $|S_n| \le \dfrac{1}{|z-z_0|^n} \dfrac{Mr_1^n(2\pi r_1)}{2\pi\,(|z-z_0|-r_1)}$

$$\le \frac{Mr_1}{(|z-z_0|-r_1)}\left(\frac{r_1}{|z-z_0|}\right)^n$$

Since $\dfrac{r_1}{|z-z_0|} < 1$, $S_n \to 0$ as $n \to \infty$.

Hence, by taking limit $n \to \infty$ in (4) we get,

$$f(z) = \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n} + \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

Hence the theorem.

Remark 9.2.2

The formulae for the coefficients a_n and b_n in the Laurent's series expansion are given by $b_n = \dfrac{1}{2\pi i}\displaystyle\int_{C_1} \frac{f(\zeta)d\zeta}{(\zeta - z_0)^{-n+1}}$ ......(1)

and $a_n = \dfrac{1}{2\pi i}\displaystyle\int_{C_2} \frac{f(\zeta)d\zeta}{(\zeta - z_0)^{n+1}}$ ....(2)

Since the integrands in the integrals of (1) and (2) are analytic functions of ζ throughout the annular region, any simple closed curve C in the annulus can be used as the path of integration in place of C_1 and C_2.

Hence Laurent's series can be written as

$$f(z) = \sum_{-\infty}^{\infty} A_n (z-z_0)^n, (r_1 < |z-z_0| < r_2) \text{ where } A_n = \frac{1}{2\pi i}\int_C \frac{f(\zeta)d\zeta}{(\zeta-z_0)^{n+1}}$$

Solved Problems

Problem 9.2.3

Find the Laurent's series expansion of $f(z) = z^2 e^{1/z}$ about $z = 0$.

Solution.

$f(z) = z^2 e^{1/z}$.

Clearly $f(z)$ is analytic at all points $z \neq 0$.

Now, $f(z) = z^2\left[1 + \dfrac{1}{z} + \dfrac{1}{2!z^2} + \dfrac{1}{3!z^3} + ...\right]$

$$= z^2 + z + \frac{1}{2} + \frac{1}{3!z} + \frac{1}{4!z^2} + ...$$

This is the required Laurent's series expansion for $f(z)$ at $z = 0$.

Space for Hints

Problem 9.2.4

Expand $\frac{-1}{(z-1)(z-2)}$ as a power series in z in the regions

i. $|z| < 1$ *ii.* $1 < |z| < 2$ *iii.* $|z| > 2$.

Solution. Let $f(z) = \frac{-1}{(z-1)(z-2)}$.

By splitting into partial fractions, we have $f(z) = \frac{1}{z-1} - \frac{1}{z-2}$

i. The only points where $f(z)$ is not analytic are 1 and 2. Hence $f(z)$ is analytic in $|z| < 1$ and hence can be represented as a Taylor's series in $|z| < 1$.

$$\therefore f(z) = \frac{1}{z-1} - \frac{1}{z-2}$$

$$= \frac{-1}{1-z} + \frac{1}{2-z}$$

$$= -(1-z)^{-1} + \frac{1}{2}\left(1 - \frac{z}{2}\right)^{-1}$$

$$= -(1 + z + z^2 + \dots + z^n + \dots) + \frac{1}{2}\left(1 + \frac{z}{2} + \frac{z^2}{4} + \dots + \frac{z^n}{2^n} + \dots\right)$$

$$= \sum_{n=0}^{\infty}\left[-z^n + \frac{1}{2}\left(\frac{z}{2}\right)^n\right]$$

$$= \sum_{n=0}^{\infty}\left(\frac{1}{2^{n+1}} - 1\right)z^n.$$

ii. $f(z)$ is analytic in the annular region $1 < |z| < 2$ and hence can be expanded as a Laurent's series in this region.

$$f(z) = \frac{1}{z-1} - \frac{1}{z-2}$$

Space for Hints

$$=\frac{1}{z\left(1-\frac{1}{z}\right)}+\frac{1}{2\left(1-\frac{z}{2}\right)}$$

$$=\frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}+\frac{1}{2}\left(1-\frac{z}{2}\right)^{-1}$$

$$=\frac{1}{z}\left[1+\left(\frac{1}{z}\right)+\left(\frac{1}{z}\right)^{2}+.....\right]+\frac{1}{2}\left(1+\frac{z}{2}+\frac{z^{2}}{4}+...+\frac{z^{n}}{2^{n}}+...\right)(\sin ce\left|\frac{1}{z}\right|<1and\left|\frac{z}{2}\right|<1$$

$$=\sum_{n=0}^{\infty}\frac{1}{z^{n+1}}+\sum_{n=0}^{\infty}\frac{z^{n}}{2^{n+1}}.$$

This gives the Laurent's series expansion in $1 < |z| < 2$.

iii. $f(z)$ is analytic in the domain $|z| > 2$ and this domain, we have, $|2/z| < 1$. Hence

$$f(z)=\frac{1}{z}\left[\frac{1}{1-(1/z)}\right]-\frac{1}{z}\left[\frac{1}{1-(2/z)}\right]$$

$$=\frac{1}{z}[1-(1/z)]^{-1}-\frac{1}{z}[1-(2/z)]^{-1}$$

$$=\frac{1}{z}\left[\left(1+\left(\frac{1}{z}\right)+\left(\frac{1}{z}\right)^{2}+.....\right)-\left(1+\left(\frac{2}{z}\right)+\left(\frac{2}{z}\right)^{2}+....\right)\right]$$

$$=\sum_{n=0}^{\infty}\frac{1-2^{n}}{z^{n+1}}.$$

Problem 9.2.5

Expand $\frac{1}{z(z-1)}$ as Laurent's series (i) about $z = 0$ in powers of z and (ii) about $z = 1$ in powers $z - 1$. Also state the region of validity.

Solution. (i) The only points where $f(z)$ is not analytic are 0 and 1. Hence $f(z)$ can be expanded as a Laurent's series in the annulus $0 < |z| < 1$.

$$f(z) = \frac{1}{z(z-1)}$$

Space for Hints

$$= -\frac{1}{z}(1-z)^{-1}$$

$$= -\frac{1}{z}(1+z+z^2+....+z^n+....) \text{ (since } |z|<1)$$

$$= -\left(\frac{1}{z}+1+z+z^2+.....+z^n+.....\right)$$

This is the Laurent's series expansion of *f(z)* in $0 < |z| < 1$

(ii) *f(z)* is analytic in $0 < |z-1| < 1$ and hence can be expanded as a Laurent's series in powers of $z-1$ in this region.

$$\frac{1}{z(z-1)} = \frac{1}{z-1}\left[\frac{1}{1+(z-1)}\right]$$

$$= \frac{1}{(z-1)}[1+(z-1)]^{-1}$$

$$= \frac{1}{z-1}[1-(z-1)+(\cdot \quad {}^2-(z-1)^3+....] \text{ (since } |z-1|<1)$$

$$= \frac{1}{z-1}-1+(z-1)-(z-1)^2+.......$$

This gives the Laurent's series expansion in $0 < |z-1| < 1$.

Problem 9.2.6

Find the Laurent's series for $\frac{z}{(z+1)(z+2)}$ about $z = -2$.

Solution. Let $f(z) = \frac{z}{(z+1)(z+2)}$

$$= \frac{-1}{z+1}+\frac{2}{z+2}(verify)$$

$$= \frac{-1}{(z+2)-1}+\frac{2}{z+2}$$

$$= [1-(z+2)]^{-1}+\frac{2}{z+2}$$

$$= [1+(z+2)+(z+2)^2+....]+\frac{2}{z+2}$$

$$= \frac{2}{z+2}+1+(z+2)+(z+2)^2+......$$

Problem 9.2.7

Expand *f(z)* = $\frac{z}{(z-1)(2-z)}$ in a Laurent's series valid for (i) $|z| < 1$

Space for Hints

(ii) $1 < |z| < 2$

(iii) $|z| > 2$ (iv) $|z - 1| > 1$ and (v) $0 < |z - 2| < 1$.

Solution.

Let $f(z) = \dfrac{z}{(z-1)(2-z)}$

$$\therefore f(z) = \frac{1}{z-1} + \frac{2}{2-z}$$ (by partial fractions)

i. $|z| < 1$.

$$f(z) = \frac{-1}{1-z} + \frac{2}{2(1-z/2)} = -(1-z)^{-1} + (1-z/2)^{-1}.$$

Since $|z| < 1$, $f(z)$ can be expanded in series as

$$f(z) = -[1 + z + z^2 + ...] + \left[1 + \left(\frac{z}{2}\right) + \left(\frac{z}{2}\right)^2 + ...\right]$$

$$= -\frac{z}{2} - \frac{3z^2}{4} - \frac{7z^3}{8} - ....$$

ii. $1 < |z| < 2$

$$f(z) = \frac{1}{z(1-1/z)} + \frac{2}{2(1-z/2)} = \frac{1}{z}(1-1/z)^{-1} + (1-z/2)^{-1}.$$

Now $1 < |z| < 2 \Rightarrow \left|\frac{1}{z}\right| < 1$ *and* $\left|\frac{z}{2}\right| < 1$. Hence we have

$$f(z) = \frac{1}{z}\left[1 + \frac{1}{z} + \left(\frac{1}{z}\right)^2 + ...\right] + \left[1 + \left(\frac{z}{2}\right) + \left(\frac{z}{2}\right)^2 + ...\right]$$

$$= ......+\left(\frac{1}{z}\right)^3 + \left(\frac{1}{z}\right)^2 + \frac{1}{z} + 1 + \left(\frac{z}{2}\right) + \left(\frac{z}{2}\right)^2 + ...$$

iii. $|z| > 2$. Hence $\left|\frac{2}{z}\right| < 1$ *and* $\left|\frac{1}{z}\right| < 1$.

$$f(z) = \frac{1}{z(1-1/z)} + \frac{2}{z(1-2/z)} = \frac{1}{z}(1-1/z)^{-1} + \frac{2}{z}(1-2/z)^{-1}.$$

Space for Hints

$$=\frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+.....\right)-\frac{2}{z}\left(1+\frac{2}{z}+\left(\frac{2}{z}\right)^2+....\right)$$

$$=-\frac{1}{z}-\frac{3}{z^2}-\frac{7}{z^3}-\frac{15}{z^4}-......$$

iv.

$|z-1|>1$. Hence $\frac{1}{|z-1|}<1$

$$f(z)=\frac{1}{z-1}-\frac{2}{z-2}.$$

$$=\frac{1}{z-1}-\frac{2}{z-1-1}.$$

$$=\frac{1}{z-1}-\frac{2}{(z-1)\left(1-\frac{1}{z-1}\right)}$$

$$=\frac{1}{z-1}-\frac{2}{z-1}\left(1-\frac{1}{z-1}\right)^{-1}$$

$$=\frac{1}{z-1}-\frac{2}{z-1}\left[1+\frac{1}{z-1}+\left(\frac{1}{z-1}\right)^2+......\right]$$

$$=-\frac{1}{z-1}-\frac{2}{(z-1)^2}-\frac{2}{(z-1)^3}-......$$

v. $0<|z-2|<1$.

$$f(z)=\frac{1}{z-2+1}-\frac{2}{z-2}.$$

$$=[1+(z-2)]^{-1}-\frac{2}{z-2}.$$

$$=[1-(z-2)+(z-2)^2-.....]-\frac{2}{z-2}$$

$$=\frac{-2}{z-2}+1-(z-2)+(z-2)^2-(z-2)^3+.......$$

Problem 9.2.8

Expand $\frac{1}{z^2-3z+2}$ in Laurent's series valid in the region $1<|z|<2$.

Solution.

Space for Hints

$$f(z)=\frac{1}{z^2-3z+2}$$

Then $f(z)=\frac{(z-2)-(z-1)}{(z-2)(z-1)}=\frac{1}{z-1}-\frac{1}{z-2}.$

f(z) is analytic in the region $1 < |z| < 2$.

Hence $f(z)$ can be expanded in Laurent's series in that region. Now

$$f(z)=\frac{1}{-2\left(1-\frac{z}{2}\right)}-\frac{1}{z\left(1-\frac{1}{z}\right)}=-\frac{1}{2}\left(1-\frac{z}{2}\right)^{-1}-\frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}.$$

In the region $1 < |z| < 2$, we have $\left|\frac{z}{2}\right|<1$ *and* $\left|\frac{1}{z}\right|<1.$ Hence $f(z)$ can be expanded in Laurent's series as

$$f(z)=-\frac{1}{2}\left[1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\left(\frac{z}{2}\right)^3+\ldots\ldots\right]-\frac{1}{z}\left[1+\frac{1}{z}+\left(\frac{1}{z}\right)^2+\left(\frac{1}{z}\right)^3+\ldots.\right]$$

$$=-\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n-\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)$$

$$=-\sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}}-\sum_{n=0}^{\infty}\frac{1}{z^{n+1}}$$

Problem 9.2.9

If $f(z)=\frac{z+4}{(z+3)(z-1)^2}$ find Laurent's series expansion in

(i) $0 < |z-1| < 4$ and (ii) $|z-1| > 4$.

Solution.

Let $f(z)=\frac{z+4}{(z+3)(z-1)^2}$

By expressing $f(z)$ into partial fractions ,we get

$$f(z)=\frac{1}{16(z+3)}-\frac{1}{16(z-1)}+\frac{5}{4(z-1)^2}.$$

(i) $0 < |z-1| < 4$. Hence $0<\left|\frac{z-1}{4}\right|<1.$

$$f(z)=\frac{1}{16(z-1+4)}-\frac{1}{16(z-1)}+\frac{5}{4(z-1)^2}.$$

Space for Hints

$$= \frac{1}{64\left(1+\frac{z-1}{4}\right)} - \frac{1}{16(z-1)} + \frac{5}{4(z-1)^2}.$$

$$= \frac{1}{64}\left(1+\frac{z-1}{4}\right)^{-1} - \frac{1}{16(z-1)} + \frac{5}{4(z-1)^2}.$$

Since $\left|\frac{z-1}{4}\right| < 1,$ we have

$$f(z) = \frac{1}{64}\left[1-\left(\frac{z-1}{4}\right)+\left(\frac{z-1}{4}\right)^2-\left(\frac{z-1}{4}\right)^3+.....\right] - \frac{1}{16(z-1)} + \frac{5}{4(z-1)^2}.$$

$$= \frac{5}{4(z-1)^2} - \frac{1}{16(z-1)} + \frac{1}{64} - \frac{1}{64}\left[\frac{z-1}{4}-\left(\frac{z-1}{4}\right)^2+.....\right].$$

This is the required Laurent's series expansion for f(z) in $0 < |z-1| < 4$.

(ii) $|z-1| > 4$. Hence $\left|\frac{4}{z-1}\right| < 1.$

Now $f(z) = \frac{1}{16(z-1)\left(1+\frac{4}{z-1}\right)} - \frac{1}{16(z-1)} + \frac{5}{4(z-1)^2}.$

$$= \frac{1}{16(z-1)}\left[1-\left(\frac{4}{z-1}\right)+\left(\frac{4}{z-1}\right)^2-....\right] - \frac{1}{16(z-1)} + \frac{5}{4(z-1)^2}.$$

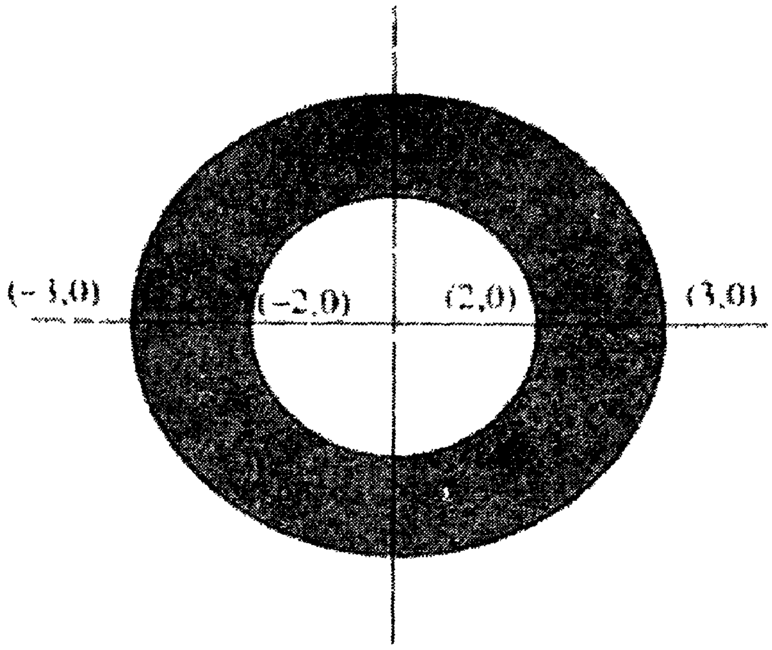
$$= \frac{1}{(z-1)^2} + \frac{1}{(z-1)^3} - \frac{4}{(z-1)^4} + \frac{4^2}{(z-1)^5} - ......$$

Problem 9.2.10

Find the Laurent's series expansion of the function $\frac{z^2-1}{(z+2)(z+3)}$ valid in the annular region $2 < |z| < 3$.

Solution.

Space for Hints



Let $f(z) = \dfrac{z^2 - 1}{(z+2)(z+3)}$.

By splitting *f(z)* into partial fractions, we get

$$f(z) = 1 + \frac{3}{z+2} - \frac{8}{z+3}.$$

f(z) is analytic in the annular region $2 < |z| < 3$.

Hence f(z) can be expanded as a Laurent's series in that region

$$f(z) = 1 + \frac{3}{z\left(1+\frac{2}{z}\right)} - \frac{8}{3\left(1+\frac{z}{3}\right)}.$$

Problem 9.2.11

For the function $f(z) = \dfrac{2z^3+1}{z(z+1)}$,

find (i) a Taylor's series valid in a neighbourhood of z = i and

(ii) a Laurent's series valid within an annulus of which centre is the origin.

Solution.

(i) $f(z) = \dfrac{2z^3+1}{z(z+1)}$

$$= 2z - 2 + \frac{1}{z} + \frac{1}{z+1} \text{ (by partial fractions)}$$

$$= 2(z-1) + \frac{1}{z} + \frac{1}{z+1} \quad \text{.......(1)}$$

$$= g(z) + h(z) + j(z).$$

Space for Hints

Where g(z) = 2(z – 1), h(z) = $\frac{1}{z}$ and j(z) = $\frac{1}{z+1}$.

Taylor's expansion for g(z) about z = i is obviously 2(i – 1) + 2(z – i).

Taylor's expansion for h(z) about z = i is given by

$$h(z) = h(i)+\sum_{n=1}^{\infty}\frac{h^{(n)}(i)}{n!}(z-i)^n.$$

Here $h(i) = \frac{1}{i}$; $h^{(n)}(z) = \frac{(-1)^n n!}{z^{n+1}}$ *so that* $h^{(n)}(i) = \frac{(-1)^n n!}{i^{n+1}}$.

$$\therefore h(z) = \frac{1}{i}+\sum_{n=1}^{\infty}\frac{(-1)^n n!}{i^{n+1}n!}(z-i)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{i^{n+1}}(z-i)^n.$$

Similarly we can prove that $j(z) = \sum_{n=0}^{\infty}\frac{(-1)^n (z-i)^n}{(1+i)^{n+1}}$

Hence the Taylor's expansion for f(z) is

$$f(z) = 2(i-1)+2(z-i)+\sum\left[\frac{(-1)^n}{i^{n+1}}+\frac{(-1)^n}{(1+i)^{n+1}}\right](z-i)^n.$$

(ii) $f(z) = 2z-2+\frac{1}{z}+(1+z)^{-1} \quad (from (1))$

$$= 2z-2+\frac{1}{z}+(1-z+z^2-z^3+.....) \quad if\ |z|<1$$

$\therefore$ In the annulus 0 < |z| < 1 the Laurent's expansion is given by

$$f(z) = \frac{1}{z}-1+z+z^2-z^3+z^4-.......$$

Problem 9.2.12

Expand $f(z) = \frac{e^{2z}}{(z-1)^3}$ about $z = 1$ as a Laurent's series. Also indicate the region of convergence of the series.

Solution.

$$f(z) = \frac{e^{2(z-1)+2}}{(z-1)^3}$$

$$= \frac{e^2 e^{2(z-1)}}{(z-1)^3}$$

$$= \frac{e^2}{(z-1)^3}\left[1 + \frac{2(z-1)}{1!} + \frac{2^2(z-1)^2}{2!} + ...\right]$$

$$= e^2\left[\frac{1}{(z-1)^3} + \frac{2}{(z-1)^2} + \frac{2}{(z-1)} + \frac{4}{3} + \frac{2^4}{4!}(z-1) + ...\right]$$

This series converges for all values of z except $z = 1$.

Space for Hints

Exercises 9.2.13

1. Expand $f(z) = \frac{1}{(z+1)(z+3)}$ in Laurent's series valid for
   (i) $1 < |z| < 3$
   (ii) $|z| > 3$
   (iii) $0 < |z+1| < 2$

Answers: 1.(i) $-\frac{1}{2z^4} + \frac{1}{2z^3} - \frac{1}{2z^2} + \frac{1}{2z} - \frac{1}{6} + \frac{z}{18} - \frac{z^2}{54} + ...$

(ii) $\frac{1}{z^2} - \frac{4}{z^3} + \frac{13}{z^4} - \frac{40}{z^5} + ...$

(iii) $\frac{1}{2(z+1)} - \frac{1}{4} + \frac{1}{8}(z+1) - \frac{1}{16}(z+1)^2 + ...$

Zeros of an Analytic Function

Definition 9.2.14

Let $f(z)$ be a function which is analytic in a region D. Let $a \in D$. Then a is said to be a ***zero of order r*** (where r is a positive integer) for $f(z)$ is $f(z) = (z-a)^r \varphi(z)$ where $\varphi(z)$ is analytic at a and $\varphi(a) \neq 0$.

Example 9.2.15

Consider $f(z) = \sin z$.

We know that $\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - .....$

$$= z\left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - .....\right)$$

$$= z\,\varphi(z).$$

Where $\varphi(z) = \left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - .....\right)$

Obviously $\varphi(z)$ is analytic and $\varphi(0) = 1 \neq 0$.

$z = 0$ is a zero of order 1 for $\sin z$.

Space for Hints

Example 9.2.16

Let $f(z) = (z-2i)^2(z+3)^3e^z$

$2i$ is a zero of order 2 and -3 is a zero of order 3 for $f(z)$.

Example 9.2.17

Let $f(z) = z^2 \sin z$.

Then $f(z) = z^2\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - .....\right)$

$= z^3\left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - .....\right)$

$= z^3\varphi(z)$

Where $\varphi(z) = \left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - .....\right)$

Obviously $\varphi(z)$ is analytic and $\varphi(0) \neq 0$.

$\therefore$

$z = 0$ is a zero of order 3 for $f(z) = z^2 \sin z$.

Example 9.2.18

Let $f(z) = \frac{z^3-1}{z^3+1}$.

$f(z) = 0 \Rightarrow z^3 - 1 = 0$

$\Rightarrow (z-1)(z^2+z+1) = 0.$

Hence the zeros of $f(z)$ are 1, $\frac{-1+i\sqrt{3}}{2}, \frac{-1-i\sqrt{3}}{2}$ and each one is a zero of order 1.

Theorem 9.2.19

Suppose $f(z)$ is analytic in a region D and is not identically zero in D. Then the set of all zeros of $f(z)$ is isolated.

Proof.

Let $a \in D$ be a zero for $f(z)$. We shall prove that there exists a neighbourhood $|z-a| < \delta$ such that this neighbourhood does not contain any other zero for $f(z)$.

Suppose a is a zero of order r for $f(z)$.

Then $f(z) = (z-a)^r\varphi(z)$ ....(1)

Where $\varphi(z)$ is analytic at a and $\varphi(a) \neq 0$.

Now, since φ is analytic at a, φ is continuous at a.

We can find a δ > 0 such that

$$|z-a| < \delta \Rightarrow |\varphi(z) - \varphi(a)| < \frac{|\varphi(a)|}{2}.$$

We claim that the neighbourhood $|z-a| < \delta$ does not contain any other zero of $f(z)$. Suppose $b \neq a$ is another zero of $f(z)$ in this neighbourhood. Then $|b-a| < \delta$ and $f(b) = 0$.

Space for Hints

$\therefore (b-a)^r \varphi(b) = 0$ (from (1))
Now, since $b \neq a$, $(b-a)^r \neq 0$
$\therefore \varphi(b) = 0$

Further $|b-a| < \delta \Rightarrow |\varphi(b) - \varphi(a)| < \dfrac{|\varphi(a)|}{2}$

$$\Rightarrow |\varphi(a)| < \frac{|\varphi(a)|}{2} \text{ which is contradiction.}$$

Thus the neighbourhood $|z-a| < \delta$ contains no other zero of $f(z)$ and hence the set of all zeros of $f(z)$ is isolated.

Corollary 9.2.20

Let $f(z)$ be analytic in a region D. Suppose $f(z) = 0$ on a subset of D which has a limit point in D. Then $f(z)$ is identically zero in D.

Corollary 9.2.21

Let $f(z)$ and $g(z)$ be two functions which are analytic in a region D. suppose $f(z) = g(z)$ on a subset of D which has a limit point in D. Then $f(z) = g(z)$ in D.
(consider the function $f(z) - g(z)$ and the result follows from corollary 9.2.20.

Exercises 9.2.21

1. Find all the zeros of the following functions

   (a) $\cos z$ (b) $\dfrac{(z+1)^2(iz+2)^3}{z+7}$

2. Prove that there is no analytic functions whose zeros are precisely the points 1, 1/2,1/3,.....,1/n,.....

Answers: 1.(a) $(2n+1)\pi/2, n\epsilon Z$ (b) -1 and $-2/i$

9.3 SINGULARITIES

Definition 9.3.1

A point a is called a ***singular point*** or ***a singularity of a function f(z)*** if $f(z)$ is not analytic at a and f is analytic at some point of every disc $|z-a| < r$.

Example 9.3.2

Consider the function $f(z) = \dfrac{1}{z}$

Then $f'(z) = -\dfrac{1}{z^2}$ for all $z \neq 0$.

Thus $f(z)$ is analytic except at z = 0.
$\therefore z = 0$ is a singular point of $f(z)$.

Space for Hints

Example 9.3.3

Consider the function $f(z) = \frac{1}{z(z-i)}$.

0 and i are singular points for $f(z)$.

Definition 9.3.4

A point a is called ***an isolated singularity for f(z)*** if

(i) $f(z)$ is not analytic at $z = a$ and

(ii) there exists $r > 0$ such that $f(z)$ is analytic in $0 < |z-a| < r$.

(i.e) the neighbourhood $|z-a| < r$ contains no singularity of $f(z)$ except a.

Example 9.3.5

$f(z) = \frac{z+1}{z^2(z^2+1)}$ has three isolated singularities $z = 0, i, -i$.

Example 9.3.6

Consider the principal branch of logarithm given by $\log re^{i\theta} = \log r + i\theta$ where $-\pi < \theta < \pi$.

All points on the negative real axis are singular points of this function. These singularities are not isolated.

Example 9.3.7

Consider the function $f(z) = \frac{1}{\sin z}$. The singular points are $0, \pm\pi, \pm2\pi$, ... and these are isolated singular points.

9.4 SINGULAR POINTS AND POLES

We now proceed to classify the isolated singularties of a function. Let a be an isolated singularity for a function $f(z)$. Let $r > 0$ be such that $f(z)$ is analytic in $0 < |z-a| < r$. In this domain the function $f(z)$ can be represented as Laurent series given by

$$f(z) = \sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n} + \sum_{n=0}^{\infty} a_n (z-a)^n \quad where$$

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(\zeta)d\zeta}{(\zeta-a)^{n+1}} \text{ and } b_n = \frac{1}{2\pi i} \int_C \frac{f(\zeta)d\zeta}{(\zeta-a)^{-n+1}}$$

The series consisting of the negative powers of $z-a$ in the above Laurent series expansion of $f(z)$ is given by $\sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n}$ and is called the ***principal part of singular part*** of $f(z)$ at $z = a$.

The singular part of $f(z)$ at $z = a$ determines the character of the singularity.

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There are three types of singularities. They are

(i) ***Removable singularities.***

(ii) ***Poles***

(iii) ***Essential singularities.***

Definition 9.4.1

Let a be an isolate singularity for $f(z)$. Then a is called a ***removable singularity*** if the principal part of $f(z)$ at $z = a$ has no terms.

Note 9.4.2

If a is a removable singularity for $f(z)$, then the Laurent's series expansion of $f(z)$ about $z = a$ is given by

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$$

$$= a_0 + a_1(z-a) + \ldots\ldots + a_n(z-a)^n + \ldots\ldots$$

Hence $\lim_{z \to a} f(z) = a_0$

Hence by defining $f(a) = a_0$ the function $f(z)$ becomes analytic at a.

Example 9.4.3

Let $f(z) = \frac{\sin z}{z}$. Clearly 0 is an isolated singular point for f(z).

Now, $\frac{\sin z}{z} = \frac{1}{z}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots\right)$

$$= 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \ldots\ldots$$

Here the principal part of $f(z)$ at $z = 0$ has no terms.

Hence $z = 0$ is a removable singularity.

Also $\lim_{z \to 0} \frac{\sin z}{z} = 1$. Hence the singularity can be removed by defining $f(0) = 1$ so that the extended function becomes analytic at $z = 0$.

Example 9.4.4

Let $f(z) = \frac{z - \sin z}{z^3}$.

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z = 0 is an isolated singularity.

Further $$\frac{z-\sin z}{z^3}=\frac{1}{z^3}\left[z-\left(z-\frac{z^3}{3!}+\frac{z^5}{5!}-\ldots\right)\right].$$

$$=\frac{1}{3!}-\frac{z^2}{5!}+\frac{z^4}{7!}-\ldots..$$

∴ $z = 0$ is a removable singularity.

By defining $f(0) = 1/6$ the function becomes analytic at $z = 0$.

Definition 9.4.5

Let a be an isolated singularity of $f(z)$. The point a is called ***a pole*** if the principal part of $f(z)$ at $z = a$ has a finite number of terms. If the principal part of $f(z)$ at $z = a$ is given by

$$\frac{b_1}{z-a}+\frac{b_2}{(z-a)^2}+\ldots..+\frac{b_r}{(z-a)^r}$$

where $b_r \neq 0$, we say that a is a ***pole of order r*** for $f(z)$.

Note 9.4.6

A pole of order 1 is called a ***simple pole*** and a pole of order of 2 is called a ***double pole.***

Example 9.4.7

Consider $$f(z)=\frac{e^z}{z}.$$

$$\frac{e^z}{z}=\frac{1}{z}+1+\frac{z}{2!}+\frac{z^2}{3!}+\ldots..$$

Here the principal part of $f(z)$ at $z = 0$ has a single term $1/z$. Hence $z = 0$ is a simple pole of $f(z)$.

Example 9.4.8

Let $f(z) = \tan z = \frac{\sin z}{\cos z}$. The singularities of $f(z)$ are $\frac{\pi}{2}+n\pi,$ where $n \in \boldsymbol{Z}$. All the singularities are poles of order 1.

Example 9.4.9

$$f(z)=\frac{\cos z}{z^2}$$ has a double pole at z = 0.

For, $$\frac{\cos z}{z^2}=\frac{1}{z^2}\left(1-\frac{z^2}{2!}+\frac{z^4}{4!}-\ldots\right)$$

$$=\frac{1}{z^2}-\frac{1}{2!}+\frac{z^2}{4!}-\ldots..$$

Example 9.4.10

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$$f(z) = \frac{z^2 - 2z + 3}{z - 2}$$

Then $f(z) = 2 + (z-2) + \frac{3}{z-2}$ *(by parital fractions)*

Here $f(z)$ has a simple pole at $z = 2$.

Definition 9.4.11

Let a be an isolated singularity of $f(z)$. The point a is called an ***essential singularity*** of $f(z)$ at $z = a$ if the principal of $f(z)$ at $z = a$ has an infinite number of terms.

Example 9.4.12

Let $f(z) = e^{1/z}$. Obviously $z = 0$ is an isolated singularity for $f(z)$.

Further $e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + ...$ The principal part of $f(z)$ has infinite number of terms. Hence $e^{1/z}$ has an essential singularity at $z = 0$.

Example 9.4.13

Let $f(z) = z^2 sin(1/z)$. $f(z)$ has essential singularity at $z = 0$.

In the following theorem , we give equivalent characterizations for an isolated singular point a of $f(z)$ to be a removable singularity.

Theorem 9.4.14

Let $f(z)$ be a function defined in a region D of the complex plane except possibly at a point $a \in D$ and let a be an isolated singularity for $f(z)$. Then a is a removable singularity for $f(z)$ if and only if there exists a complex number a_0 such that by defining $f(a) = a_0$ the extended function becomes analytic at a.

Proof. Suppose a is a removable singularity for f(z).

Then $f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n, 0 < |z-a| < r$

$= a_0 + a_1(z-a) + a_2(z-a)^2 + \ldots\ldots$

$\therefore$ By defining $f(z) = a_0$, f becomes analytic at a.

Conversely, suppose there exists a complex number a_0 such that by defining $f(a) = a_0, f$ becomes analytic in $|z-a| < r$.

Hence f can be represented as a Taylor's series, in power of $z - a$ in this neighbourhood, given by $f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$.

This shows that the principal of $f(z)$ at $z = a$ has no terms. Hence a is a removable singularity for $f(z)$.

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Theorem 9.4.15 (Riemann's Theorem)

Let f be a function which is bounded and analytic throughout a domain $0 < |z - z_0| < \delta$. Then either f is analytic at z_0 or else z_0 is a removable singular point of f.

Proof. Consider the Laurent's series for the function in the given domain about z_0. The coefficient b_n of $\frac{1}{(z-z_0)^n}$ is given by

$$b_n = \frac{1}{2\pi i}\int_C \frac{f(z)dz}{(z-z_0)^{-n+1}}$$ where C is the circle $|z - z_0| = r$ where $r < \delta$.

Now, since f is bounded there exists a positive real number M such that $|f(z)| \leq M$ in $0 < |z - z_0| < \delta$.

by theorem $\left|\int_C f(z)dz\right| \leq Ml$ *where* $M = \max\{|f(z)| : z \in C\}$, *we have,*

$$\therefore |b_n| \leq \frac{1}{2\pi}\frac{M(2\pi r)}{r^{-n+1}}$$
$$= Mr^n$$

Since it is true for every r such that $0 < r < \delta$, taking limit $r \to 0$ we get $b_n = 0$. Hence the Laurent's series for $f(z)$ has no principal part. Hence the theorem follows.

Theorem 9.4.16

Let $f(z)$ be a function having a as an isolated singular point. Then the following are equivalent.

(i) a is a pole of order r for $f(z)$.

(ii) $f(z)$ can be written in the form $f(z) = \frac{1}{(z-a)^r}\theta(z)$ where $\theta(z)$ has a removable singularity at $z = a$ and and $\lim_{z\to a}\theta(z) \neq 0$.

(iii) a is a zero of order r for $\frac{1}{f(z)}$.

Proof. $(i) \Rightarrow (ii)$

Let a be a pole order r for $f(z)$. Then the Laurent's series expansion of $f(z)$ about a is given by $f(z) = \sum_{n=1}^{r}\frac{b_n}{(z-a)^n} + \sum_{n=0}^{\infty}a_n(z-a)^n$ where $b_r \neq 0$.

$$\therefore f(z) = \frac{1}{(z-a)^r}\left[b_r + b_{r-1}(z-a) + \dots + b_0(z-a)^{r-1} + a_0(z-a)^r + \dots\right]$$

$$= \frac{1}{(z-a)^r}\theta(z) \text{ where } \theta(z) = b_r + b_{r-1}(z-a) + \dots$$

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Clearly $\lim_{z\to a}\theta(z) = b_r \neq 0$ and $\theta(z)$ has a removable singularity at $z = a$.

$(ii) \Rightarrow (iii)$ Let $f(z) = \frac{1}{(z-a)^r}\theta(z)$ and by suitably defining θ(a) we may assume that θ(z) is analytic at a and θ(a) ≠ 0.

$\frac{1}{f(z)} = (z-a)^r \frac{1}{\theta(z)}$ *and* $\frac{1}{\theta(z)}$ *is analytic at a and* $\frac{1}{\theta(a)} \neq 0.$

Hence a is a zero of order r for $\frac{1}{f(z)}$.

$(iii) \Rightarrow (i)$ Let a be a zero of order r for $\frac{1}{f(z)}$.

Then $\frac{1}{f(z)} = (z-a)^r g(z)$ where $g(z)$ is analytic at a and $g(a) \neq 0$.

$\therefore f(z) = \frac{g_1(z)}{(z-a)^r}$ where $g_1(z)$ is analytic at a and $g_1(a) \neq 0$.

Let $g_1(z) = a_0 + a_1(z-a) + \ldots + a_n(z-a)^n + \ldots$ so that $a_0 \neq 0$.

$$\therefore f(z) = \frac{a_0}{(z-a)^r} + \frac{a_1}{(z-a)^{r-1}} + \ldots + a_r + a_{r+1}(z-a) + \ldots in\, 0 < |z-a| < r$$

$\therefore$The principal part of f(z) at $z = a$ is

$$\frac{a_0}{(z-a)^r} + \frac{a_1}{(z-a)^{r-1}} + \ldots + \frac{a_r}{z-a} \text{ and } a_0 \neq 0.$$

$\therefore$ a is a pole of order r for $f(z)$.

Theorem 9.4.17

An isolated singularity a of $f(z)$ is a pole if and only if $\lim_{z\to a} f(z) = \infty$.

Proof. If a is a pole of order r for $f(z)$ then $f(z) = \frac{g(z)}{(z-a)^r}$ with g(a) ≠ 0.

$\therefore \lim_{z\to a} f(z) = \infty$.

Conversely let a be an isolated singularity for $f(z)$ and let $\lim_{z\to a} f(z) = \infty$.

Let $\theta(z) = \frac{1}{f(z)}$.

Then $\lim_{z\to a}\theta(z) = 0$.

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Hence a is a removable singularity for $\theta(z)$ and by defining $\theta(z) = 0$, θ becomes analytic at a. Let a be a zero of order r for the function $\theta(z)$. Then a is a pole of order r for $f(z)$.

Definition 9.4.18

A function $f(z)$ is said to be a ***meromorphic function*** if it is analytic except at a finite number of points and these finite set of points are poles.

Example 9.4.19

Let $f(z) = \dfrac{1}{z(z-1)^2}$.

$f(z)$ is analytic except at $z = 0$ and $z = 1$. Also 0 and 1 are poles of order 1 and 2 respectively. He‚ ‵z) is a meromorphic function.

Example 9.4.20

$\dfrac{e^z}{z} = \dfrac{1}{z} + 1 + \dfrac{z}{2!} + \dfrac{z^2}{3!} + ...$ is a meromorphic function.

Example 9.4.21

$e^{1/z}$ is not a meromorpic function since $z = 0$ is an essential singularity for $e^{1/z}$.

The following theorem due to Weierstrass describes the behavior of a function in the neighbourhood of an essential singularity.

Theorem 9.4.22

Let z_0 be an essential singularity for a function $f(z)$. Let c be any complex number. Then given ε, $\delta > 0$ there exists a point z such that $|z - z_0| < \delta$ and $|f(z) - c| < \varepsilon$.

(i.e) The function $f(z)$ comes arbitrarily close to any complex number c in every neighbourhood of an essential singularity.

Proof. Suppose the theorem is false. Then there exist δ, $\varepsilon > 0$ such that for every point z satisfying $0 < |z - z_0| < \delta$ we have $|f(z) - c| \geq \varepsilon$.

Now consider the function $g(z) = \dfrac{1}{f(z) - c}$.

$$\therefore |g(z)| = \frac{1}{|f(z) - c|} \leq \frac{1}{\varepsilon}.$$

Hence $g(z)$ is bounded and further $g(z)$ is analytic in $0 < |z - z_0| < \delta$.
Hence by Riemann's theorem, $z = z_0$ is a removable singularity for $g(z)$.

Now, if $g(z_0) \neq 0$ then $\frac{1}{g(z)} = f(z) - c$ is analytic at z_0.

Therefore by suitably defining $g(z_0)$, the function $g(z)$ becomes analytic at z_0.

If $g(z_0) = 0$ then let z_0 be a zero of order r for g(z).

Then z_0 is a pole of order r for $\frac{1}{g(z)} = f(z) - c$.

Thus $f(z)$ is either analytic at z_0 or else z_0 is a pole of $f(z)$ which is a contradiction to the hypothesis that z_0 *is* an essential singularity for $f(z)$.
Hence the theorem.

Solved Problems

Problem 9.4.23

Determine and classify the singular points of $f(z) = \frac{e^z}{e^z - 1}$.

Solution.

The singularities of $f(z)$ are given by the values of z for which $e^z - 1 = 0$.
Hence $z = 2n\pi i$, $n \in \mathbf{Z}$, are the singularities of $f(z)$.

$$Now, e^z - 1 = \left(1 + z + \frac{z^2}{2!} + ... + \frac{z^n}{n!} + ...\right) - 1.$$

$$= z + \frac{z^2}{2!} + ... + \frac{z^n}{n!} + ...$$

$$\therefore \lim_{z \to 0} \frac{z}{e^z - 1} = 1$$

Hence 0 is a removable singularity for $f(z)$.

Also $\lim_{z \to 2n\pi} \left(\frac{z}{e^z - 1}\right) = \infty$ if $n \neq 0$ and hence $2n\pi i$, $n \neq 0$, are simple poles of f(z).

Problem 9.4.24

Determine and classify the singularities of $f(z) = sin(1/z)$.

Solution.

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Also

$$f(z)=\frac{1}{z}-\frac{1}{3!z^3}+\frac{1}{5!z^5}-.....$$

Thus the principal part of *f(z)* at *z = 0* has infinitely many terms and hence 0 is an essential singularity for *f(z)*.

Problem 9.4.25

Determine and classify the singular points of $\frac{1}{(2\sin z-1)^2}$.

Solution.

The singularities of *f(z)* are given by the values of z for which

2 sin *z – 1 = 0.*

The singularities of f(z) are given by $z=\frac{\pi}{6}+2n\pi, n\in Z,$ and they are double poles.

Exercises 9.4.26

1. Find the singularities of the following functions and classify the singularities.

   (i) $\frac{z}{e^{1/z}-1}$ (ii) $(z-i)\sin\left(\frac{1}{z+2i}\right)$

   (iii) $\frac{z^2-2z+3}{z-2}$ (iv) $ze^{1/z}$

2. Show that the singular points of each of the following functions are poles. Determine the order of each pole.

   (i)*tanh z* (ii) $\frac{e^{2z}}{(z-1)^2}$ (iii) $\frac{1}{z^2(z-3)^2}$

   (iv) $\frac{z+1}{z^2-2z}$ (v) $\frac{z(1+z)}{1-\cos z}$ (vi) $\frac{1}{z^4+2z^2+1}$

3. Find the order of the pole z = 0 for the following functions.

   (i) $\frac{e^z}{z}$ (ii) $\frac{e^z}{z^2}$ (iii) $\frac{1-\sin z}{z^5}$

Answes: 2. (i) 0 is a simple pole (ii) 1 is a double pole (iii) 0,3 are double poles (iv) 0 and 2 are simple poles (v) 0 is a simple pole (vi) *i* and *–i* are double poles.

3. (i)1 (ii) 2 (iii) 5

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UNIT 10

CALCULUS OF RESIDUES

In this chapter, we introduce the concept of the residue of a function *f(z)* at an isolated singular point and prove Cauchy's residue therorem. Using this theorem we evaluate certain types of real definite integrals.

10.1 RESIDUES

Definition 10.1.1

Let *a* be an isolated singularity for *f(z)*. Then the ***residue of f(z)*** at *a* is defined to be the coefficient of $\frac{1}{z-a}$ in the Laurent's series expansion of *f(z)* about *a* and is denoted by *Res{f(z);a}*.

Thus $\operatorname{Res}\{f(z);a\} = \frac{1}{2\pi i}\int_C f(z)dz = b_1$ Where C is a circle $|z-a|$ =

such that is analytic in $0 < |z-a| < r$.

Example 10.1.2

Consider $f(z) = \frac{e^z}{z^2}$

$$\frac{e^z}{z^2} = \frac{1}{z^2}\left(1 + \frac{z}{1!} + \frac{z^2}{2!} + \ldots\ldots\right)$$

$$= \frac{1}{z^2} + \frac{1}{z} + \frac{1}{2!} + \frac{z}{3!} + \frac{z^2}{4!} + \ldots\ldots$$

Therefore *f(z)* has a double pole at $z = 0$.

Therefore *Res{f(z) ; 0 }* = coefficient of $1/z = 1$.

The following lemmas provide methods for calculation of residues.

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Lemma 10.1.3

If $z = a$ is a simple pole for $f(z)$, then
$\text{Res}\{f(z);a\} = \lim_{z \to a}(z-a)f(z)$.

Proof.

Since $z = a$ is a simple pole for $f(z)$, the Laurent's series expansion

for $f(z)$ about $z = a$ is given by

$$f(z) = \frac{b_1}{z-a} + a_0 + a_1(z-a) + \ldots\ldots$$

Now, $(z-a)f(z) = b_1 + a_0(z-a) + a_1(z-a)^2 + \ldots\ldots$

$\therefore \lim_{z \to a}(z-a)f(z) = b_1$

$= Res\{f(z);\ a\}$.

Lemma 10.1.4

If a is a simple pole for $f(z)$ and $f(z) = \frac{g(z)}{z-a}$ where $g(z)$ is analytic at a and $g(a) \neq 0$, then $Res\{f(z);a\} = g(a)$.

Proof.

By Lemma 10.1.3, $\text{Res}\{f(z);a\} = \lim_{z \to a}(z-a)f(z)$
$= \lim_{z \to a} g(z) = g(a)$.

Lemma 10.1.5

If a is simple pole for $f(z)$ and if $f(z)$ is of the form $\frac{h(z)}{k(z)}$ where $h(z)$ and $k(z)$ are analytic at a and $h(a) \neq 0$ and $k(a) = 0$, then
$\text{Res}\{f(z);a\} = \frac{h(a)}{k'(a)}$.

Proof.

$$= \lim_{z\to a}(z-a)\frac{h(z)}{k(z)}.$$

$$= \lim_{z\to a} h(z) \lim_{z\to a}\frac{(z-a)}{k(z)}.$$

$$= \lim_{z\to a} h(z) \lim_{z\to a}\left[\frac{z-a}{k(z)-k(a)}\right]. \quad \text{(since } k(a) = 0\text{)}$$

$$= h(a)\left[\frac{1}{k'(a)}\right]$$

$$\operatorname{Res}\{f(z);a\} = \frac{h(a)}{k'(a)}$$

Lemma 10.1.6

Let a be a pole or order m > 1 for $f(z)$ and let $f(z) = \frac{g(z)}{(z-a)^m}$ where $g(z)$ is analytic at a and $g(a) \neq 0$.

Then $\operatorname{Res}\{f(z);a\} = \frac{g^{(m-1)}(a)}{(m-1)!}$.

Proof.

$$g^{(m-1)}(a) = \frac{(m-1)!}{2\pi i}\int_C \frac{g(z)dz}{(z-a)^m} \text{ (by theorem on higher derivatives)}$$

where C is a circle $|z-a| = r$ such that $f(z)$ is analytic in $0 < |z-a| < r$.

$$\therefore \quad \frac{g^{(m-1)}(a)}{(m-1)!} = \frac{1}{2\pi i}\int_C f(z)dz = \operatorname{Res}\{f(z);a\}.$$

SOLVED PROBLEMS

Problem 10.1.7

Calculate the residue of $\frac{z+1}{z^2-2z}$ at its poles.

Solution.

Let $f(z) = \frac{z+1}{z^2-2z} = \frac{z+1}{z(z-2)}$.

$z = 0$ and $z = 2$ are simple poles for $f(z)$.

$$\operatorname{Res}\{f(z);0\} = \lim_{z\to 0}(z-0)\left[\frac{z+1}{z(z-2)}\right]$$

$$= \lim_{z\to 0}\frac{z+1}{z-2} = -\frac{1}{2}$$

$$\operatorname{Res}\{f(z);2\} = \lim_{z\to 2}(z-2)\left[\frac{z+1}{z(z-2)}\right]$$

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$$= \lim_{z \to 2} \frac{z+1}{z} = \frac{3}{2}.$$

Problem 10.1.8

Find the résidue at $z = 0$ of $\frac{1+e^z}{z \cos z + \sin z}$.

Solution.

Let $f(z) = \frac{1+e^z}{z \cos z + \sin z}$.

Clearly 0 is a pole of order 1 for $f(z)$.

$\text{Re}\, s\{f(z);0\} = \lim_{z \to 0} \frac{h(z)}{k'(z)}$ where $h(z) = 1 + e^z$ and

$k(z) = z \cos z + \sin z$, by Lemma 10.1.5.

Now, $k'(z) = -z\sin z + \cos z + \cos z$

$= -z\sin z + 2\cos z$

Therefore, $\text{Re}\, s\{f(z);0\} = \frac{2}{2} = 1$

Problem 10.1.9

Use Laurent's series to find the residue of $\frac{e^{2z}}{(z-1)^2}$. at z = 1.

Solution.

Let f(z) = $\frac{e^{2z}}{(z-1)^2}$.

First we expand $f(z)$ as Laurent's series at $z = 1$.

$$f(z) = \frac{e^{2(z-1)+2}}{(z-1)^2}.$$

$$= \frac{e^2 e^{2(z-1)}}{(z-1)^2}.$$

$$= \frac{e^2}{(z-1)^2}.\left[1 + \frac{2(z-1)}{1!} + \frac{2^2(z-1)^2}{2!} + \frac{2^3(z-1)^3}{3!} + \dots\dots\right]$$

$$= e^2.\left[\frac{1}{(z-1)^2} + \frac{2}{z-1} + 2 + \frac{4}{3}(z-1) + \dots\dots\right]$$

This is Laurent's series expansion for $f(z)$ at $z = 1$.

$\text{Re}\, s\{f(z);1\}$ = coefficient of $\frac{1}{z-1}$. in Laurent's expansion

$= 2e^2$.

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Note 10.1.10
Without expanding in Laurent's series the residue at z = 1 can be found as follows. Since *f(z)* has a pole of order 2 at $z = 1$, we choose $g(z) = e^{2z}$.

$$\therefore \quad \operatorname{Res}\{f(z);1\} = \frac{g'(1)}{1!} = \left[\frac{2e^{2z}}{1!}\right]_{z=1} = 2e^2.$$

Problem 10.1.11

Find the residue of $\dfrac{ze^z}{(z-1)^3}$ at its pole.

Solution.

Let $f(z) = \dfrac{ze^z}{(z-1)^3}$.

$z = 1$ is a pole of order 3 for *f(z)*.
Let $g(z) = ze^z$ so that $g'(z) = e^z(z+1)$ and $g''(z) = e^z(z+2)$.

Then $\operatorname{Res}\{f(z);1\} = \dfrac{g''(z)}{2!} = \dfrac{3e}{2}$.

Problem 10.1.12

Find the residue of $\dfrac{1}{z - \sin z}$ at its pole.

Solution.

Let $f(z) = \dfrac{1}{z - \sin z}$

Now $z - \sin z = z - \left(z - \dfrac{z^3}{3!} + \dfrac{z^5}{5!} - \dots\dots\dots\right)$

$$= \frac{z^3}{3!} - \frac{z^5}{5!} + \dots\dots\dots$$

$$= z^3\left(\frac{1}{3!} - \frac{z^2}{5!} + \dots\dots\dots\right)$$

$z = 0$ is a pole of order 3 for *f(z)* and $f(z) = \dfrac{1}{z^3\left(\dfrac{1}{3!} - \dfrac{z^2}{5!}\dots\dots\right)}$

Now let $g(z) = \dfrac{1}{\left(\dfrac{1}{3!} - \dfrac{z^2}{5!}\dots\dots\right)}$

Then $Res\ \{f(z);\ 0\} = \dfrac{g''(0)}{2!}$. Clearly $g(0) = 6$.

Now $\dfrac{1}{g(z)} = \dfrac{1}{3!} - \dfrac{z^2}{5!} + \dfrac{z^4}{7!} - \dots\dots$ Differentiating with

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respect to z, we have $-\frac{g'(z)}{[g(z)]^2} = -\frac{2z}{5!} + \frac{4z^3}{7!} - .....$

Hence g'(0) = 0.

Again differentiating with respect to z we have,

$$\frac{[g(z)]^2[-g''(z)] + g'(z)2g(z)g'(z)}{[g(z)^4]} = \frac{-2}{5!} + \frac{12z^2}{7!} - ....$$

Putting $z = 0$ and using $g(0) = 6$ and $g'(0) = 0$ we get $\frac{-g''(0)}{36} = \frac{-2}{5!}$

Hence g''(0) = $\frac{3}{5}$

Therefore Res {f(z); 0 } = $\frac{g''(0)}{2!} = \frac{3}{10}$.

Exercises 10.1.13

1. Find the order of each and find the residue at the poles for each of the following functions.

(i) $\frac{z}{(z^2+1)}$ (ii) $\frac{z+1}{z^2-2z}$

(iii) $\frac{2z+3}{z(z^2+1)}$ (iv) $\frac{z^2}{z^2+a^2}$

2. Find the residue of $\frac{z^2-2z}{(z+1)^2(z^2+4)}$ at all its poles.

3. Find the residue of $\frac{1}{(1+z^2)^n}$ at z = i.

Answers: 1. (i)Simple pole;Res ½;*-i* simple pole;Res *1/2*

(ii)*z=0,2* simple pole;Res-*1/2,3/2*

(iii) *0,i,-i* simple poles;Res *3,(2i+3)/2,(2i-3)/2*

(iv) *ai,-ai* simple poles;Res *ai/2,-ai/2*

2. z=-1;-14/25;z=2i;(7+i)/25;(7-i)/25.

3. $\frac{-i(2n-2)!}{(2^{2n-1}[(n-1)!]^2}$

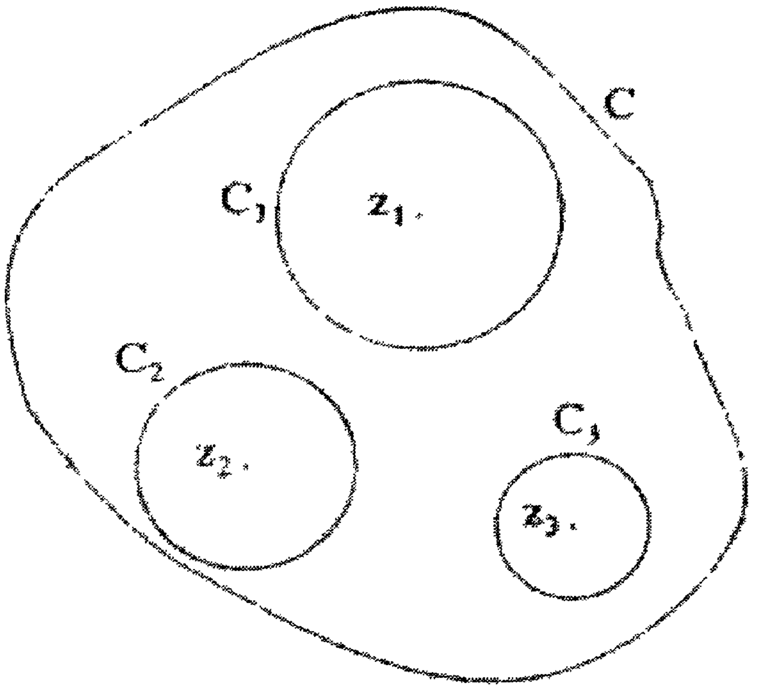
Theorem 10.1.14(Cauchy's Residue Theorem)

Let *f(z)* be a function which is analytic inside and on a simple closed curve *C* except for a finite number of singular points $z_1, z_2, ,........,z_n$ inside *C*.

Then $\int_C f(z)dz = 2\pi i \sum_{j=1}^{n} Res\{f(z); z_j\}$

Proof.

Space for Hints



Let $C_1, C_2, C_3, \ldots\ldots\ldots C_n$ be circles with centres $z_1, z_2, \ldots$ $\ldots\ldots z_n$ respectively such that all circles are interior to C and are disjoint with each other.(refer figure).

By Cauchy's theorem for multiply connected regions, we have

$$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz + \ldots\ldots\ldots + \int_{C_n} f(z)dz.$$

$= 2\pi i\, Res\{f(z);z_1\} + 2\pi i\, Res\{f(z);z_2\} + \ldots\ldots\ldots\ldots + 2\pi i\, Res\{f(z);z_n\}$
(by definition of residue)

$$= 2\pi i \sum_{j=1}^{n} \operatorname{Re} s\{f(z); z_j\}.$$

Hence the theorem.

Example 10.1.15

Evaluate $\int_C \frac{z^2}{(z-2)(z+3)}$ where C is the circle $|z| = 4$.

Let $f(z) = \frac{z^2}{(z-2)(z+3)}$

$z=2$ and $z=-3$ are simple poles for $f(z)$ and both of them lie inside $|z| = 4$.

Now, Res $\{f(z) ; 2\} = \lim_{z\to 2}(z-2)\left[\frac{z^2}{(z-2)(z+3)}\right] = \frac{4}{5}$

Res $\{f(z) ; -3\} = \lim_{z\to -3}(z+3)\left[\frac{z^2}{(z-2)(z+3)}\right] = -\frac{9}{5}$

Therefore by residue theorem,

Space for Hints

$$\int_C f(z)dz = 2\pi i\left[\frac{4}{5}+\left(-\frac{9}{5}\right)\right]$$

$= - 2\pi$ i

$$\therefore \int_C \frac{z^2}{(z-2)(z+3)} = -2\pi i$$

Theorem 10.1.16 (Argument Theorem)

Let f be a function which is analytic inside and on a simple closed curve C except for a finite number of poles inside C. Also let $f(z)$ have no zeros on C. Then $\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz = N - P$ where N is the number of zeros of $f(z)$ inside C and P is the number of poles of $f(z)$ inside C.

(A pole or zero of order m is counted m times).

Proof.

We observe that the singularities of the function $\frac{f'(z)}{f(z)}$ inside C are the poles and zeros of order n for $f(z)$. Let C_1 be a circle with centre z_0 such that it is the only zero of $f(z)$ inside C_1. Then $f(z) = (z-z_0)^n g(z)$ where $g(z)$ is analytic and nonzero inside C_1. Hence $f'(z) = n(z-z_0)^{n-1} g(z) + (z-z_0)^n g'(z)$

$$\therefore \frac{f'(z)}{f(z)} = \frac{n}{z-z_0} + \frac{g'(z)}{g(z)} \text{ -- } (1)$$

Since $g(z)$ is analytic and non zero inside C_1. $\frac{g'(z)}{g(z)}$ is also analytic and hence can be expanded as a Taylor's series about z_0.

$$\therefore \quad \operatorname{Res}\{\frac{f'(z)}{f(z)}; z_0\} = coefficient\ of\ \frac{1}{z-z_0} in (1)$$

Similarly if z_1 is a pole of order p for $f(z)$, then $\operatorname{Res}\{\frac{f'(z)}{f(z)}; z_1\} = -p$.

Hence by Cauchy's residue theorem, $\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz = \text{N} - \text{P}$ where N is the number of zeros and P is the number of poles of $f(z)$ within C.

Corollary 10.1.17

If $f(z)$ is analytic inside and on C and not zero on C, then $\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz = \text{N}$ Where N is the number of zeros lying inside C.

Proof.

Since the number of poles is zero, we have, P = 0 .

Hence the result follows.

Space for Hints

Theorem 10.1.18 (Rouche's Theorem)

If $f(z)$ and $g(z)$ are analytic inside and on a simple closed curve C and if $|g(z)| < |f(z)|$ on C, then $f(z) + g(z)$ and $f(z)$ have the same number of zeros inside C.

Proof.

$$f(z)+g(z) = f(z)\left[1+\frac{g(z)}{f(z)}\right] = f(z)\,\phi(z) \text{ where } \phi(z) = \left[1+\frac{g(z)}{f(z)}\right].$$

Hence $[f(z)+g(z)]' = f'(z) + g'(z) = f'(z)\,\phi(z) + f(z)\,\phi'(z)$

$$\therefore \quad \frac{f'(z)+g'(z)}{f(z)+g(z)} = \frac{f'(z)\varphi(z)+f(z)\varphi'(z)}{f(z)\varphi(z)}$$

$$= \frac{f'(z)}{f(z)} + \frac{\varphi'(z)}{\varphi(z)}$$

$$\therefore \quad \frac{1}{2\pi i}\int_C \left[\frac{f'(z)+g'(z)}{f(z)+g(z)}\right]dz = \frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz$$

$$+ \frac{1}{2\pi i}\int_C \frac{\varphi'(z)}{\varphi(z)}dz \text{ ----------(1)}$$

Now, by hypothesis $|g(z)| < |f(z)|$ and hence $\left|\frac{g(z)}{f(z)}\right| < 1$ on C.

Therefore $|\varphi(z) - 1| < 1$ on C.

Hence by maximum modulus theorem, $|\varphi(z) - 1| < 1$ for every point z inside C.

Therefore $\varphi(z) \neq 0$ for every point inside C.

Hence $\int_C \frac{\varphi'(z)}{\varphi(z)}dz$ = Number of zeros of $\varphi(z)$ within C.

$= 0.$

Hence from (1) we have, $\frac{1}{2\pi i}\int_C \left[\frac{f'(z)+g'(z)}{f(z)+g(z)}\right]dz = \frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz$

Therefore $N_1 = N_2$ where N_1 and N_2 denote respectively the number of zeros of $f(z) + g(z)$ and $f(z)$ inside C. Hence the theorem.

Remark 10.1.19

We can deduce the Fundamental Theorem of Algebra from Rouche's theorem.

Theorem 10.1.20 (Fundamental Theorem of Algebra)

A Polynomial of degree n with complex coefficients has n zeros in C.

Proof.

Let $a_0 + a_1 z + a_2 z^2 + \ldots\ldots\ldots + a_n z^n$, where $a_n \neq 0$, be a polynomial of degree m.

Let $f(z) = a_n z^n$ and $g(z) = a_0 + a_1 + \ldots\ldots + a_{n-1} z^{n-1}$

Space for Hints

Clearly $\lim\limits_{z\to\infty}\dfrac{g(z)}{f(z)} = 0.$

Hence there exists a positive real number r such that $\left|\dfrac{g(z)}{f(z)}\right| < 1$

for all z with $|z| > r$.

Hence by Rouche's theorem $f(z)$ and $f(z) + g(z)$ have the same number of zeros inside the circle $|z| = r + 1$. But 0 is a zero of multiplicity n for $f(z)$.

Hence the gives polynomial $f(z) + g(z)$ also has n zeros.

Solved Problems

Problem1 10.1.21

Evaluate $\displaystyle\int_C \frac{dz}{2z+3}$ where C is $|z| = 2$.

Solution.

$z = -\dfrac{3}{2}$ is the simple pole of $f(z)$ which lies inside the circle $|z| = 2$.

$\operatorname{Res}\{f(z); -\frac{3}{2}\} = \lim\limits_{z\to -3/2}\dfrac{h(z)}{k'(z)}$ where $h(z) = 1$ and $k(z) = 2z + 3$.

$\operatorname{Res}\{f(z); -\frac{3}{2}\} = \dfrac{1}{2}$.

Therefore by residue theorem, $\displaystyle\int_C f(z)dz = 2\pi i\left(\frac{1}{2}\right) = \pi i$

Problem 10.1.22

Evaluate $\displaystyle\int_C \frac{dz}{z^2e^z}$ where $C = \{z; |z| = 1\}$.

Solution.

Given integral can be written as $\displaystyle\int_C f(z)dz$ where $f(z) = \dfrac{e^{-z}}{z^2}$

$f(z)$ has pole of order 2 at $z = 0$ which lies inside the circle $|z| = 1$.

Let $g(z) = e^{-z}$. Hence $g'(z) = -e^{-z}$.

Therefore by Lemma 10.1.6, $\operatorname{Res}\{f(z);0\} = \dfrac{g'(0)}{1!} = -1$.

By residue theorem, $\displaystyle\int_C f(z)dz = 2\pi i(-1) = -2\pi i$.

Problem 10.1.23

Evaluate $\displaystyle\int_C \frac{2+3\sin\pi z}{z(z-1)^2}dz$ where C is the square having vertices $3+3i$, $3-3i$, $-3+3i$, $-3-3i$.

Solution.

Space for Hints

Let $f(z) = \dfrac{2+3\sin\pi z}{z(z-1)^2}$. Here $z = 0$ is a simple pole and $z = 1$ is a double pole for $f(z)$ and both of them lie within C.

$$Res\ \{f(z)\ ;\ 0\} = \lim_{z\to 0}(z)\left(\frac{2+3\sin\pi z}{z(z-1)^2}\right) = 2.$$

$$\mathrm{Re}\,s\{f(z);1\} = \frac{g'(1)}{1!} \quad \text{where} \quad g(z) = \frac{2+3\sin\pi z}{z}.$$

$$g'(z) = \frac{z3\pi\cos\pi z - (2+3\sin\pi z)}{z^2}$$

Therefore $g'(1) = -3\pi - 2$.

Therefore $\mathrm{Re}\,s\{f(z);1\} = -3\pi - 2$.

Therefore by residue theorem, $\int_C f(z)dz = 2\pi\, i(2-3\pi-2) = -6\pi^2\, i$.

Problem 10.1.24

Evaluate $\int_C \tan z dz$ where C is $|z| = 2$.

Solution.

Let $f(z) = \tan z = \dfrac{\sin z}{\cos z} = \dfrac{h(z)}{k(z)}$

$Cos\ z$ has zeros at $z = \dfrac{(2n+1)\pi}{2}, n \in N$.

Therefore $f(z)$ has simple poles at $z = -\dfrac{\pi}{2}$ and $z = \dfrac{\pi}{2}$ inside the circle $|z| = 2$.

$$\mathrm{Re}\,s\{f(z);\pi/2\} = \frac{h(\pi/2)}{k'(\pi/2)} = \frac{\sin(\pi/2)}{-\sin(\pi/2)} = -1$$

$$\mathrm{Re}\,s\{f(z);-\pi/2\} = \frac{h(-\pi/2)}{k'(-\pi/2)} = \frac{\sin(-\pi/2)}{-\sin(-\pi/2)} = -1$$

By residue theorem, $\int_C \tan z dz = 2\pi\, i\,[(-1)+(-1)] = -4\pi\, i$.

Problem 10.1.25

Prove that $\int_C \dfrac{e^{2z}}{(z+1)^3}dz = \dfrac{4\pi i}{e^2}$ where C is $|z| = \dfrac{3}{2}$.

Solution.

Let $f(z) = \dfrac{e^{2z}}{(z+1)^3}$

$f(z)$ has a pole of order 3 at $z = -1$.

Space for Hints

$\operatorname{Re}s\{f(z);-1\} = \dfrac{g''(-1)}{2!}$ where $g(z) = e^{2z}$.

Now $g'(z) = 2e^{2z}$ and $g''(z) = 4e^{2z}$.

$\operatorname{Re}s\{f(z);-1\} = \dfrac{4e^{-2}}{2!} = \dfrac{2}{e^2}$

Therefore by residue theorem , $\int_C f(z)dz = 2\pi i\left(\dfrac{2}{e^2}\right) = \dfrac{4\pi i}{e^2}$

Problem 10.1.26

Evaluate, using (i) Cauchy's integral formula (ii) residue theorem

$\int_C \dfrac{z+1}{z^2+2z+4}dz$ where C is the circle $|z+1+i| = 2$.

Solution.

Clearly C is a circle with centre $a = -(1+i)$ and radius 2.

Now $\dfrac{z+1}{z^2+2z+4} = \dfrac{z+1}{(z+1)^2 \ldots)^2}$

$= \dfrac{z+1}{(z+1+i\sqrt{3})\ \ +(z+1-i\sqrt{3})}$

$= \dfrac{z+1}{[z-(-1-i\sqrt{3}]\ \ +[z-(-1+i\sqrt{3}]}$

$z_0 = -1+i\sqrt{3}$ *and* $z_1 = -1-i\sqrt{3}$ are the singular points of the given integrand

$\dfrac{z+1}{z^2+2z+4}$.

Now $|z_0 - a| = |\, i(\sqrt{3}+1)|\ | = \sqrt{3}+1 > 2$

and $|z_1 - a| = |-i(\sqrt{3}-1)|\ | = \sqrt{3}-1 < 2$

$\therefore\ z_1 = -1-i\sqrt{3}$ lies inside C.

(i) By using Cauchy integral formula.

Consider $f(z) = \dfrac{z+1.}{[z-(-1-i\sqrt{3}]}$

We note that f(z) is analytic at all points inside C.

Therefore by Cauchy's integral formula, $\dfrac{1}{2\pi i}\int_C \dfrac{f(z)}{z-z_1}dz = f(z_1)$;

(i.e) $\dfrac{1}{2\pi i}\int_C \dfrac{(z+1)dz}{[z-(1-i\sqrt{3})][z-(-1+i\sqrt{3})]} = f(-1-i\sqrt{3})$

(*i.e*) $\dfrac{1}{2\pi i}\int_C \dfrac{(z+1)dz}{z^2+2z+4} = \dfrac{(-1-i\sqrt{3})+1}{(-1-i\sqrt{3})-(-1+i\sqrt{3})}$

Space for Hints

$$= \frac{-i\sqrt{3}}{-2i\sqrt{3}} = \frac{1}{2}$$

$$\therefore \int_C \frac{z+1}{z^2+2z+4}dz = \frac{1}{2}(2\pi i) = \pi i$$

(ii) By using residue theorem,

$$f(z) = \frac{z+1}{z^2+2z+4}$$

$\because$ $z = -1 - i\sqrt{3}$ lies inside C.

$\operatorname{Re} s\{f(z); -1-i\sqrt{3}\} = \frac{h(-1-i\sqrt{3})}{k'(-1-i\sqrt{3})}$ where $h(z) = z+1$ and

$k(z) = z^2 + 2z + 4$ so that $k'(z) = 2z+2$

$$\operatorname{Re} s\{f(z); -1-i\sqrt{3}\} = \frac{-1-i\sqrt{3}+1}{2(-1-i\sqrt{3})+2} = \frac{-i\sqrt{3}}{-i2\sqrt{3}} = \frac{1}{2}$$

By residue theorem, $\int_C f(z)dz = \frac{2\pi i}{2} = \pi i.$

Problem 10.1.27

Use residue calculus to evaluate $\int_C \frac{3\cos z}{2i-3z}dz$ where C is the unit circle.

Solution.

Let $f(z) = \frac{3\cos z}{2i-3z}$.

Here $z = \frac{2i}{3}$ is a simple pole and lies within C.

$\operatorname{Re} s\{f(z); \frac{2i}{3}\} = \lim_{z\to 2i/3} \frac{h(z)}{k'(z)}$ where $h(z) = 3\cos z$ and

$k(z) = 2i - 3z$ so that $k'(z) = -3$.

$\operatorname{Re} s\{f(z); \frac{2i}{3}\} = \frac{3\cos(2i/3)}{-3} = -\cos(2i/3) = -\cosh(2/3).$

By residue theorem $\int_C f(z)dz = 2\pi i.[-\cosh(2/3)]$

(i.e) $\int_C \frac{3\cos z}{2i-3z}dz = -2\pi i.[\cosh(2/3)]$.

Problem 10.1.28

Use residue theorem to evaluate $\int \frac{3z^2+z-1}{(z^2-1)(z-3)}dz$ around the circle $|z| = 2$.

Solution.

Space for Hints

Let $f(z) = \dfrac{3z^2 + z - 1}{(z^2 - 1)(z - 3)}$

$f(z)$ has simple poles 1, -1 , 3 and only 1, -1 lie inside $|z| = 2$.

$\operatorname{Re} s\{f(z);1\} = \dfrac{h(1)}{k'(1)}$ where $h(z) = 3z^2 + z - 1$ and

$k(z) = z^3 - 3z^2 - z + 3$ so that

$k'(z) = 3z^2 - 6z - 1$.

$$\operatorname{Re} s\{f(z);1\} = \frac{3+1-1}{3-6-1} = \frac{-3}{4}$$

Similarly $\operatorname{Re} s\{f(z);-1\} = \dfrac{3-1-1}{3+6-1} = \dfrac{1}{8}$

Therefore by residue theorem,

$$\int \frac{3z^2 + z - 1}{(z^2 - 1)(z - 3)} dz == (2\pi i)\left(\frac{-3}{4} + \frac{1}{8}\right) = 2\pi i\left(\frac{-5}{8}\right) = \frac{-5\pi i}{4}$$

Problem 10.1.29

Evaluate $\displaystyle\int_C \frac{e^z}{(z-1)(z+2)} dz$ where C is the circle $|z-1| = 1$.

Solution.

$$f(z) = \frac{e^z}{(z+2)((z-1)}$$

$f(z)$ has simple poles at 1, -2; the pole 1 is inside the circle $|z-1| = 1$ and $z = -2$ lies outside the circle.

$$\operatorname{Re} s\{f(z);1\} = \lim_{z \to 1} (z-1)\left(\frac{e^z}{(z+2)(z-1)}\right) = \frac{e}{3}$$

By residue theorem, $\displaystyle\int_C f(z) dz = 2\pi i.\left(\frac{e}{3}\right)$.

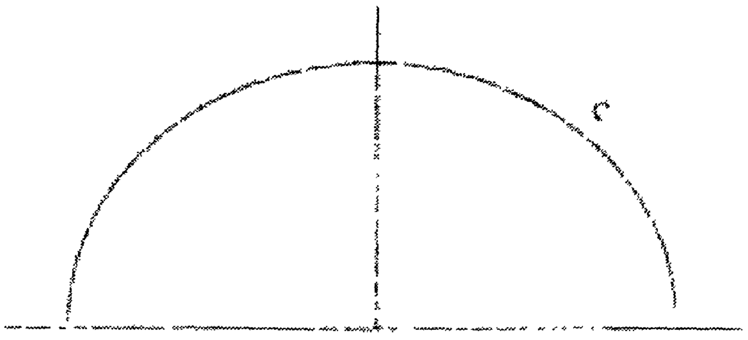
$$\therefore \int_C \frac{e^z}{(z-1)(z+2)} dz = \frac{i2\pi e}{3}.$$

Problem 10.1.30

Show that the function $2 + z^2 - e^{iz}$ has precisely one zero in the open upper half plane.

Solution.

Space for Hints



Take $f(z) = 2 + z^2$ and $g(z) = - e^{iz}$. Let C be the simple closed curve consisting of the semi circle $|z| = r$ in the upper half plane together with the interval $[-r,r]$ on the real axis.

If $z \in [-r,r]$, then $|g(z)| = 1$ and $|f(z)| \geq 1$.

Hence $|f(z)| > |g(z)|$.

Now, if $z = re^{i\theta}$, $0 < \theta < \pi$, then $|f(z)| = |2+z^2| \geq |z^2| - 2 = r^2 - 2$.

Also $|g(z)| = \left|- e^{ire^{i\theta}}\right| = e^{-r\sin\theta}$.

Hence for sufficiently large value of r, we have $|f(z)| > |g(z)|$.
Hence by Rouche's theorem, $f(z) + g(z) = 2 + z^2 - e^{iz}$ and $f(z) = 2 + z^2$ have the same number of zeros in the upper half plane. Also $2+z^2$ has exactly one zero in the upper half of the plane namely $i\sqrt{2}$.

Hence $2 + z^2 - e^{iz}$ has exactly one root in the upper half plane.

Problem 10.1.31

Let $f(z) = \dfrac{(z^2+1)}{(z^2+2z+z)^2}$ .Evaluate $\dfrac{1}{2\pi i}\displaystyle\int_C \dfrac{f'(z)}{f(z)}dz$ where C is the circle $|z| = 4$.

Solution.

I and – I are zeros of order 1 and -1 +I and -1 –I are poles of order 2 for $f(z)$. Also these zeros and poles lie inside C.
Hence number of zeros of $f(z) = N = 2$ and number of poles of $f(z)$ = $= 4$.(Poles are counted according to their multiplicity)

Therefore by Argument theorem , $\dfrac{1}{2\pi i}\displaystyle\int_C \dfrac{f'(z)}{f(z)}dz = N - P = 2 - 4$
-2.

Exercises 10.1.32

1.Evaluate the following integrals

(*i*) $\displaystyle\int_C \dfrac{3z-4}{z(z-1)}dz$ where C is the circle $|z|=2$.

Space for Hints

(ii) $\int_C \frac{3z-4}{z(z-1)(z-2)} dz$ where C is the circle $|z|=3/2$.

(iii) $\int_C \frac{3dz}{(z+1)}$ where C is the circle $|z|=2$.

(iv) $\int_C \frac{3+z}{z} dz$ where C is the circle $|z|=1$.

Answers: 1.(i) $6\pi i$ (ii) $-2\pi i$ (iii) $-2\pi i$ (iv) $6\pi i$

10.2 EVALUATION OF DEFINITE INTEGRALS

We use Cauchy's residue theorem for evaluation certain types of real definite integrals.

Type 1

$\int_0^{2\pi} f(\cos\theta, \sin\theta) d\theta$ where $f(\cos\theta, \sin\theta)$ is a rational function of $\cos\theta$ and $\sin\theta$.

To evaluate this type of integral we substitute $z = e^{i\theta}$. As θ varies from 0 to 2π, z describes the unit circle $|z| = 1$.

Also, $\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{z + z^{-1}}{2}$. And

$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{z - z^{-1}}{2i}$

Substituting these values in the given integrand the integral is transformed into

$\int_C \theta(z) dz$ where $\theta(z) = f\left[\frac{z + z^{-1}}{2}, \frac{z - z^{-1}}{2i}\right]$ and C is the positively oriented unit circle $|z| = 1$.

The integral $\int_C \theta(z) dz$ can be evaluated using the residue theorem.

Solved Problems

Problem 10.2.1

Evaluate $\int_0^{2\pi} \frac{d\theta}{5 + 4\sin\theta}$

Solution.

Let I = $\int_0^{2\pi} \frac{d\theta}{5 + 4\sin\theta}$

Put $z = e^{i\theta}$

Then $dz = iz\, d\theta$ and $\sin\theta = \frac{z - z^{-1}}{2i}$

Space for Hints

The given integral is transformed to I = $\int_C \frac{dz}{iz\left[5+4\left(\frac{z-z^{-1}}{2i}\right)\right]}$

where C is the unit circle $|z| = 1$.

$$= \int_C \frac{dz}{2z^2+5iz-2}$$

Let $f(z) = \frac{1}{2z^2+5iz-2} = \frac{1}{2(z+2i)(z+i/2)}$

Therefore $-2i$ and $-i/2$ are simple poles of $f(z)$ and the pole $-i/2$ lies inside C.

Also $\text{Res}\{f(z); \frac{-i}{2}\} = \lim_{z \to i/2} \frac{1}{2(z+2i)} = \frac{1}{3i}$

Hence by Cauchy's residue theorem, I = 2 πi $\left(\frac{1}{3i}\right) = \frac{2\pi}{3}$.

Problem 10.2.2

Prove that $\int_0^{2\pi} \frac{d\theta}{1+a\sin\theta} = \frac{2\pi}{\sqrt{1-a^2}}$,$(-1 < a < 1)$

Solution.

Put $z = e^{i\theta}$. then $\sin\theta = \frac{z-z^{-1}}{2i}$ and $dz = iz\, d\theta$.

$\int_0^{2\pi} \frac{d\theta}{1+a\sin\theta} = \int_C \frac{dz}{iz\left[1+a\left(\frac{z-z^{-1}}{2i}\right)\right]}$ where C is the unit circle .

$$= \int_C \frac{dz}{z[2i+a(z-z^{-1})]}$$

$$= \int_C \frac{2dz}{az^2+2iz-a}$$

Let $f(z) = \frac{2}{az^2+2iz-a}$

The poles of $f(z)$ are given by $z = \frac{-2i \pm \sqrt{-4+4a^2}}{2a}$

Space for Hints

$$= \frac{-i \pm i\sqrt{1-a^2}}{a} \text{ (since } -1 < a < 1);$$

Let $z_1 = \dfrac{-i + i\sqrt{1-a^2}}{a}$ and $z_2 = \dfrac{-i - i\sqrt{1-a^2}}{a}$

We note that $|z_2| = \dfrac{1+\sqrt{1-a^2}}{|a|} > 1$ (since $-1 < a < 1$);

Also, since $|z_1 z_2| = 1$ it follows that $|z_1| < 1$. Hence there are no singular points on C and $z = z_1$ is the only simple pole inside C.

$$\operatorname{Res}\{f(z); z_1\} = \lim_{z \to z_1} (z - z_1)\left[\frac{2/a}{(z-z_1)(z-z_2)}\right]$$

$$= \left[\frac{2/a}{z_1 - z_2}\right]$$

$$= \frac{1}{i\sqrt{1-a^2}}$$

By residue theorem, $\displaystyle\int_0^{2\pi} \frac{d\theta}{1 + a\sin\theta} = 2\pi i\left[\frac{1}{i\sqrt{1-a^2}}\right]$

$$= \frac{2\pi}{\sqrt{1-a^2}}$$

Problem 10.2.3

Prove that $I = \displaystyle\int_0^{\pi} \frac{a\,d\theta}{a^2 + \sin^2\theta} = \frac{\pi}{\sqrt{a^2+1}}$ (a>0)

Solution.

$$I = \int_0^{\pi} \frac{a\,d\theta}{a^2 + \left(\dfrac{1-(\cos 2\theta}{2}\right)}$$

$$= \int_0^{\pi} \frac{2a d\theta}{2a^2 + 1 - \cos 2\theta}$$

$$= \int_0^{2\pi} \frac{a d\varphi}{2a^2 + 1 - \cos\varphi} \qquad \text{(putting } 2\theta = \varphi\text{)}$$

$$= \frac{1}{i}\int_C \frac{a dz}{z\left[2a^2 + 1 - \frac{(z + z^{-1})}{2}\right]} \qquad \text{(putting } z = e^{i\varphi}\text{)}$$

$$= \frac{2a}{i}\int_C \frac{dz}{[2(2a^2+1)z - z^2 - 1]}$$

$$= 2ai\int_C \frac{dz}{[-2(2a^2+1)z + z^2 + 1]}$$

$$= 2ai\int_C f(z)dz \quad \text{----------------------------(1)}$$

Where $f(z) = \dfrac{1}{z^2 - 2(2a^2+1)z + 1}$ and C is the unit circle $|z| = 1$.

Poles of $f(z)$ are the roots of $z^2 - 2(2a^2+1)z + 1 = 0$.

Therefore $z = (2a^2+1) \pm 2a\sqrt{a^2+1}$

Let $z_1 = (2a^2+1) + 2a\sqrt{a^2+1}$; $z_2 = (2a^2+1) - 2a\sqrt{a^2+1}$

Clearly $|z_1| > 1$ and $|z_1 z_2| = 1$ so that $|z_2| < 1$.

Hence the only pole inside C is $z = z_2$.

$$\operatorname{Res}\{f(z); z_2\} = \lim_{z \to z_2} (z - z_2)\left[\frac{1}{(z - z_1)(z - z_2)}\right]$$

$$= \left[\frac{1}{z_2 - z_1}\right]$$

Space for Hints

Space for Hints

$$= \frac{1}{(-4a)\sqrt{a^2+1}}$$

From (1), I $= 2\pi i\left[\dfrac{2ai}{-4a\sqrt{a^2+1}}\right]$

$$= \frac{\pi}{\sqrt{1-a^2}}$$

Problem 10.2.4

Using Contour integration, evaluate $\int_0^{2\pi}\frac{d\theta}{13+5\sin\theta}$

Solution.

Let I = $\int_0^{2\pi}\frac{d\theta}{13+5\sin\theta}$

Put $z = e^{i\theta}$. Then $\sin\theta = \frac{z-z^{-1}}{2i}$ and $dz = iz\,d\theta$.

Therefore the given integral is transformed to

I = $\int_C \frac{dz}{iz\left[13+5\left(\frac{z-z^{-1}}{2i}\right)\right]}$ (where C is the circle $|z| = 1$)

$$= \int_C \frac{dz}{iz\left[13+5\left(\frac{z^2-1}{i2z}\right)\right]}$$

$$= \int_C \frac{2dz}{5z^2+26iz-5} \quad \ldots$$

Let $f(z) = \dfrac{2}{5z^2+26iz-5} = \dfrac{2}{(z+5i)(5z+i)}$

$\frac{-i}{5}$ and $-5i$ are simple poles of $f(z)$ and the pole $\frac{-i}{5}$ lies inside the unit circle.

$\operatorname{Re}s\{f(z);-\frac{i}{5}\} = \lim\limits_{z\to -i/5}\frac{h(z)}{k'(z)}$ (where *h(z)* = *2* and

k(z) = 5z2 + i26z - 5).

$$= \lim_{z\to -i/5}\left(\frac{2}{-2i+26i}\right)$$

$$= \left(\frac{2}{-2i+26i}\right)$$

$$= \frac{1}{12i}$$

Hence by Cauchy's residue theorem, I $= 2\pi i\left(\frac{1}{12i}\right) = \frac{\pi}{6}$

Problem 10.2.5

Use Contour integration technique to find the value of $\int_0^{2\pi}\frac{d\theta}{2+\cos\theta}$.

Solution.

Let I = $\int_0^{2\pi}\frac{d\theta}{2+\cos\theta}$

Put $z = e^{i\theta}$. Then $\sin\theta = \frac{z-z^{-1}}{2i}$ and $dz = iz\, d\theta.$

Also $\cos\theta = \frac{z+z^{-1}}{2}$

The given integral is transformed to $I = \int_C \frac{dz}{iz\left[2+\left(\frac{z+z^{-1}}{2}\right)\right]}$

(where *C* is the circle $|z| = 1$)

$$\therefore I = \int_C \frac{dz}{iz\left[2+\left(\frac{z^2+1}{2z}\right)\right]}$$

$$= \int_C \frac{-2idz}{(4z \;+z^2+1)}$$

Space for Hints

Let $f(z) = \dfrac{-2i}{z^2+4z+1} = \dfrac{-2i}{(z+2)^2-3} = \dfrac{-2i}{(z+2-\sqrt{3})(z+2+\sqrt{3})}$

$\therefore$ $-2+\sqrt{3}$ and $-2-\sqrt{3}$ are simple poles of $f(z)$ and the pole $-2+\sqrt{3}$ lies inside C.

$\operatorname{Res}\{f(z);-2+\sqrt{3}\} = \lim\limits_{z\to-2+\sqrt{3}}\left[\dfrac{-2i}{2z+4}\right]$

$= \left(\dfrac{-2i}{-4+2\sqrt{3}+4}\right)$

$= \dfrac{-i}{\sqrt{3}}$

Hence by Cauchy's residue theorem I $= 2\pi i\left(\dfrac{-i}{\sqrt{3}}\right) = \dfrac{2\pi}{\sqrt{3}}$

Exercises 10.2.6

1.Show that

(i) $\displaystyle\int_0^{2\pi}\frac{d\theta}{5+3\cos\theta} = \frac{\pi}{2}$

(ii). $\displaystyle\int_0^{2\pi}\frac{d\theta}{5+4\cos\theta} = \frac{\pi}{6}$

(iii). $\displaystyle\int_0^{2\pi}\frac{d\theta}{2+\cos\theta} = \frac{2\pi}{\sqrt{3}}$.

Type 2

$\displaystyle\int_{-\infty}^{\infty} f(x)dx$ where $f(x) = \dfrac{g(x)}{h(x)}$ and $g(x)$, $h(x)$ are polynomials in z and the degree of $h(x)$ exceeds that of $g(x)$ by at least two.

To evaluated this type of integral, we take $f(z) = \dfrac{g(z)}{h(z)}$.

The poles of $f(z)$ are determined by the zeros of the equation $h(z) = 0$.

Case (i) No pole of $f(z)$ liles on the real axis.

We choose the curve C consisting of the interval $[-r,r]$ on the real axis and the semi circle $|z| = r$ lying in the upper half of the plane. Here r is chosen sufficiently large so that all the poles lying in the upper half of the plane are in the interior of C. Then we have

$\displaystyle\int_C f(z)dz = \int_{-r}^{r} f(x)dx + \int_{C_1}^{\infty} f(z)dz$ where C_1 is the semi circle.

Since $deg\ h(x) - deg\ f(x) \geq 2$ it follows that and hence

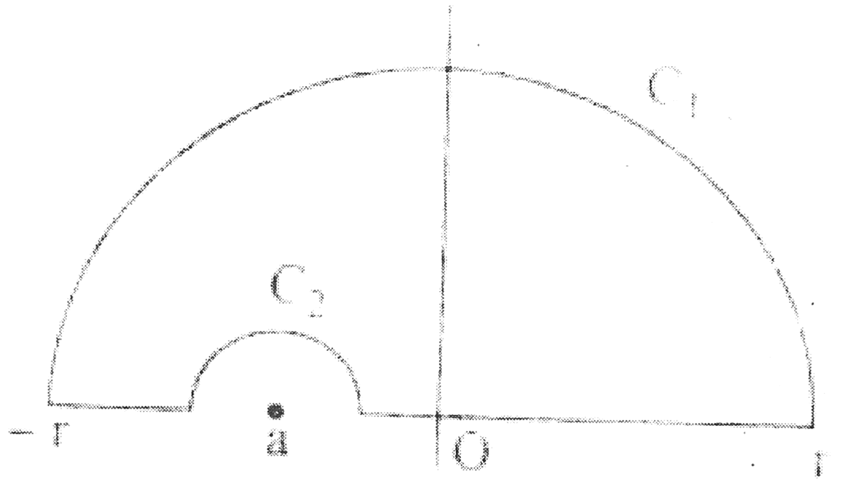
Space for Hints

$$\int_C f(z)dz = \int_{-\infty}^{\infty} f(x)dx$$

$\therefore \int_{-\infty}^{\infty} f(x)dx$ can be evaluated by evaluating $\int_C f(z)dz$ which in turn can be evaluated by using Cauchy's residue theorem.

Case (ii) *f(z)* has poles lying on the real axis .

Suppose a is a pole lying on the real axis. In this case we indent the real axis by a semi- circle C_2 of radius ε with centre a lying in the upper half plane where ε is chosen to be sufficiently small (refer figure).



Such an indenting must be done for every pole of *f(z)* lying on the real axis.

it can be proved that $\int_{C_2} f(z)dz$ = - πi Res{f(z);a}. By taking limit as $r \to \infty$

and $\varepsilon \to 0$, we obtain the valued of $\int_{-\infty}^{\infty} f(x)dx$.

Solved Problems

Problem 10.2.7

Use Contour integration method to evaluate $\int_0^{\infty} \frac{dx}{1+x^4}$

Solution.

Let $f(z) = \dfrac{1}{1+z^4}$

The poles of *f(z)* are given by the roots of the equation $z^4 + 1 = 0$, which are the four fourth roots of -1.

By De Moivre's theorem, they are given by $e^{i\pi/4}$; $e^{i3\pi/4}$;$e^{i5\pi/4}$; $e^{i7\pi/4}$ and all are simple poles.

We choose the contour C consisting of the interval $[-r,r]$ on the

Space for Hints

real axis and the upper semi-circle $|z| = r$ which we denote by C_1.

$$\therefore \int_C f(z)dz = \int_{-r}^{r} f(x)dx + \int_{C_1} f(z)dz \text{ ------------------------(1)}$$

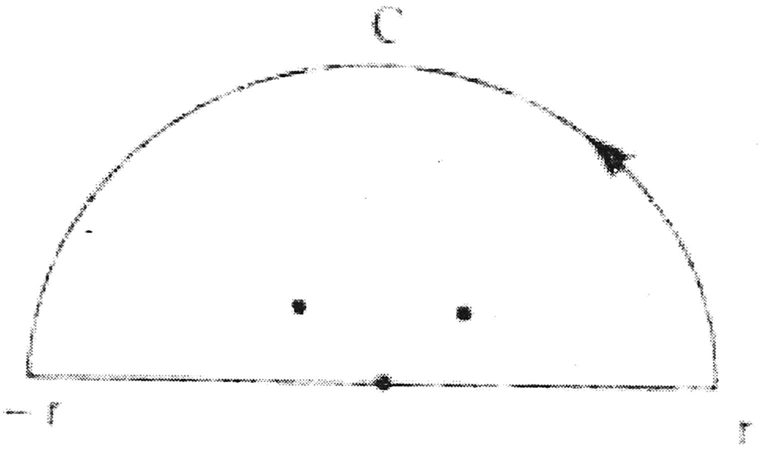
The poles of $f(z)$ lying inside the contour C are obviously $e^{i\pi/4}$ and $e^{i3\pi/4}$ only.

We find the residues of $f(z)$ at these points.

$$\operatorname{Res}\{f(z);e^{i\pi/4}\} = \frac{h(e^{i\pi/4})}{k'(e^{i\pi/4})}$$ where $h(z) = 1$ and $k(z) = z^4 + 1$

so that $k'(z) = 4z^3$.

$$\operatorname{Res}\{f(z);e^{i\pi/4}\} = \frac{1}{4e^{i3\pi/4}} = \frac{e^{-i3\pi/4}}{4}$$



Similarly $\operatorname{Res}\{f(z);e^{i3\pi/4}\} = \dfrac{e^{-i9\pi/4}}{4}$

By residue theorem, $\int_C f(z)dz = 2\pi i$(sum of the residues at the poles).

$$=2\pi i\left[\frac{e^{-i3\pi/4}}{4} + \frac{e^{-i9\pi/4}}{4}\right]$$

$$= \frac{\pi i}{2}\left[(\cos(3\pi/4) - i\sin(3\pi/4) + (\cos(9\pi/4) - i\sin(9\pi/4)\right]$$

$$= \frac{\pi i}{2}\left[\left(\frac{-1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)\right]$$

$$= \frac{\pi i}{2}\left(\frac{-2i}{\sqrt{2}}\right) = \frac{\pi}{\sqrt{2}}$$

From (1), $\displaystyle\int_{-r}^{r} \frac{dx}{1+x^4} + \int_{C_1} f(z)dz = \frac{\pi}{2}$

As $r \to \infty$, $\int_{C_1} f(z)dz \to 0$.

$$\therefore \int_0^\infty \frac{dx}{1+x^4} = \frac{\pi}{\sqrt{2}}$$

$$\therefore 2\int_0^\infty \frac{dx}{1+x^4} = \frac{\pi}{\sqrt{2}} \quad (\because \frac{1}{1+x^4} \text{ is an even function})$$

$$\therefore \int_0^\infty \frac{dx}{1+x^4} = \frac{\pi}{2\sqrt{2}}$$

Problem 10.2.8

Using the method of contour integration, evaluate

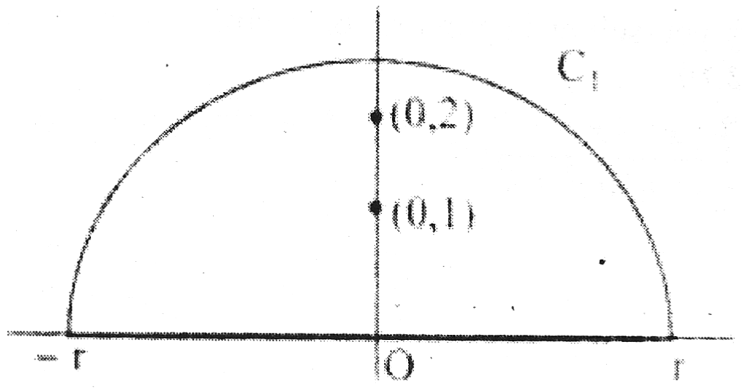
$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+4)} dx.$$

Solution.

Let $f(z) = \dfrac{z^2}{(z^2+1)(z^2+4)}$

The poles of $f(z)$ are $i, -i, 2i, -2i$.

Choose the contour C as shown in the figure.



The poles i and $2i$ lie within C. By residue theorem,

$$\int_C f(z)dz = 2\pi i \text{ (sum of the residues of } f(z) \text{)-----------------(1)}$$

We find the residues of $f(z)$.

$\operatorname{Res}\{f(z);i\} = \dfrac{h(i)}{k'(i)}$ where $h(z) = z^2$ and

$k(z) = (z^2+1)(z^2+4) = z^4 + 5z^2 + 4$ so that $k'(z) = 4z^3 + 10z$.

Space for Hints

$$\operatorname{Re}s\{f(z);i\} = \frac{-1}{-4i+10i} = \frac{-1}{6i} = \frac{i}{6}.$$

$$\operatorname{Re}s\{f(z);2i\} = \frac{-i}{3} \qquad (verify).$$

$$\therefore \quad From\,(1) \quad \int_C f(z)dz = 2\pi\, i\left(\frac{i}{6} - \frac{i}{3}\right)$$

$$= 2\pi\, i\left(\frac{-i}{6}\right)$$

$$= \left(\frac{\pi}{3}\right) \text{-----------------------------------(2)}$$

Therefore also (1) can be written, using (2) , as

$$\int_{C_1} f(z)dz + \int_{-r}^{r} \frac{x^2}{(x^2+1)(x^2+4)}dx = \frac{\pi}{3} \quad \text{-----------------------------------(3)}$$

Further the integral $\int_{C_1} f(z)dz \to 0$ as $r \to \infty$.

$$\therefore \quad \int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+4)}dx = \frac{\pi}{3}.$$

Problem 10.2.9

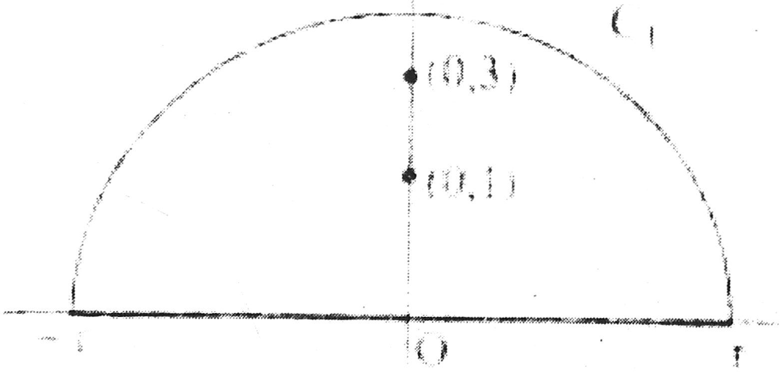
Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{(x^2+a^2)(x^2+b^2)}dx = \frac{\pi}{a+b}$.

Solution. Try yourself as in previous problem.

Problem 10.2.10

Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{x^2 - x + 2}{x^4 + 10x^2 + 9}dx.$

Solution.



Space for Hints

Let $f(z) = \dfrac{z^2 - z + 2}{z^4 + 10z^2 + 9}$.

Poles of $f(z)$ are the zeros of $z^4+10z^2+9 = 0$.

$z^4+10z^2+9 = (z^2+9)(z^2+1)$.

Therefore $z = \pm 3i$, $\pm i$. Hence $z = 3i$, $-3i$, i, $-i$ are the simple poles of $f(z)$.

Choose the contour C as shown in the figure.

$$\int_C f(z)dz = \int_{-r}^{r} f(x)dx + \int_{C_1} f(z)dz \quad ....(1)$$

The poles of $f(z)$ lying within C are i and $3i$ and both of them are simple poles.

$\text{Re}\,s\{f(z);i\} = \dfrac{h(i)}{k'(i)}$ where $h(z) = z^2-z+2$ and $k(z) = z^4 + 10z^2 + 9$

so that $k'(z) = 4z^3+20z$.

$$\text{Re}\,s\{f(z);i\} = \frac{-1-i+2}{-4i+20i} = \frac{1-i}{16i}$$

$$\text{Re}\,s\{f(z);3i\} = \frac{7+3i}{48i} \quad (verify).$$

$$\therefore \int_C dz = 2\pi i(\text{sum of the residues at the poles})$$

$$= 2\pi i\left(\frac{1-i}{16i} + \frac{7+3i}{48i}\right)$$

$$= 2\pi i\left(\frac{10}{48i}\right) = \frac{5\pi}{12}$$

From (1), $$\int_{-r}^{r} \frac{x^2 - x + 2}{x^4 + 10x^2 + 9}dx + \int_{C_1} \frac{z^2 - z + 2}{z^4 + 10z^2 + 9} = \frac{5\pi}{12}$$

Now as $r \to \infty$ the integral over $C_1 \to 0$.

Therefore, $$\int_{-\infty}^{\infty} \frac{x^2 - x + 2}{x^4 + 10x^2 + 9}dx = \frac{5\pi}{12}.$$

Problem 10.2.11

Evaluate $I = \displaystyle\int_0^{\infty} \frac{dx}{(x^2 + a^2)^2}$.

Solution.

Since $\dfrac{1}{(x^2 + a^2)^2}$ is an even function, we have

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2 + a^2)^2} = 2\int_0^{\infty} \frac{dx}{(x^2 + a^2)^2}.$$

Space for Hints

Let $f(z) = \dfrac{1}{(z^2+a^2)^2}$

Poles of $f(z)$ are the roots of $(z^2+a^2)^2 = 0$.
Now, $(z^2+a^2)^2 = (z+ai)^2\,(z-ai)^2$.
Therefore ai and $-ai$ are double poles of $f(z)$.
Choose the contour C consisting of the interval $[-r, r]$ on the real axis and the semi circle C_1 with centre 0 and radius r that lies in the upper half plane.

$$\therefore \int_C f(z)dz = \int_{-r}^{r} f(x)dx + \int_{C_1} f(z)dz \text{ ------(1)}$$

The poles of $f(z)$ lying within C is $z = ai$.

$\operatorname{Res}\{f(z);ai\} = \dfrac{1}{1!}g'(ai)$ where $g(z) = \dfrac{1}{(z+ai)^2}$

Now $g'(z) = -2\,(z+ai)^{-3}$.

Therefore $g'(ai) = \dfrac{1}{4a^3 i}$

Therefore,

$$\operatorname{Res}\{f(z);ai\} = \frac{1}{4a^3 i}.$$

$$\therefore \int_C f(z)dz = 2\pi i\left(\frac{i}{4a^3 i}\right) = \frac{\pi}{2a^3}.$$

$$\therefore \int_{-r}^{r}\frac{dx}{(x^2+a^2)^2} + \int_{C_1} f(z)dz = \frac{\pi}{2a^3}.$$

When as $r \to \infty$, the integral over $C_1 \to 0$.

$$\therefore \int_{-\infty}^{\infty}\frac{dx}{(x^2+a^2)^2} = \frac{\pi}{2a^3}.$$

$$\therefore \int_{0}^{\infty}\frac{dx}{(x^2+a^2)^2} = \frac{\pi}{a^3}.$$

Problem 10.2.12

Prove that $\displaystyle\int_0^{\infty}\frac{x^4dx}{(x^6-1)} = \frac{\pi\sqrt{3}}{6}.$

Solution.

Space for Hints

Now let $f(z)= \dfrac{z^4}{(z^6-1)}$.

The poles of $f(z)$ are given by the sixth roots of unity, namely $e^{2n\pi i/6}$; $n = 0, 1, \ldots\ldots, 5$.

Therefore $f(z)$ has 2 simple poles on the real axis , viz., 1 and -1 and two poles $e^{\pi i/3}$ and $e^{2\pi i/3}$ lie on the upper half of the plane.

$$\therefore \int_C f(z)dz = \int_{C_1} f(z)dz + \int_{-r}^{-1-\varepsilon_1} f(x)dx + \int_{C_2} f(z)dz + \int_{-1+\varepsilon_1}^{-1-\varepsilon_1} f(x)dx +$$

$$\int_{C_3} f(z)dz \qquad \int_{1+\varepsilon_2}^{r_1} f(x)dx \text{-------}(1)$$

Now, $\int_{C_2} f(z)dz = -\pi$ i Res {f(z) ; -1}

$$= \quad -\pi i\left(\frac{h(-1)}{k'(-1)}\right) \text{ where } h(z) = z^4$$

$= -\pi$ i(-1/6)

$= \pi$ i/6.-----------(2)

Similarly $\int_{C_3} f(z)dz = -\pi$ i Res {f(z) ; 1}

$$= \quad -\pi i\left(\frac{h(1)}{k'(1)}\right)$$

$= -\pi\ i(1/6)$

$= -\pi\ i/6$.---------(3)

Also $\int_C f(z)dz = 2\pi i\ [\ Res\ \{f(z)\ ;\ e^{\pi i/3}\} + Res\ \{f(z)\ ;\ e^{2\pi i/3}\}]$

$$= \quad 2\pi i\left[\frac{h(e^{i\pi/3})}{6e^{i5\pi/3}} + \frac{e^{i8\pi/3}}{6e^{i10\pi/3}}\right]$$

$$= \quad 2\pi i\left[\frac{e^{i4\pi/3}}{6e^{i5\pi/3}} + \frac{e^{i8\pi/3}}{6e^{i10\pi/3}}\right]$$

$$= \quad \frac{\pi i}{3}\left(e^{-i\pi/3} + e^{-i2\pi/3}\right)$$

$$= \quad \frac{\pi i}{3}\left(e^{-i\pi/3} - e^{i\pi/3}\right)$$

$$= \quad \frac{\pi i}{3}\left(-2i \sin \pi/3\right)$$

$$= \quad \frac{\pi\sqrt{3}}{3} \text{----------}(4)$$

Space for Hints

Substituting (2), (3), (4) in (1) and taking limits as $\epsilon_1, \epsilon_2 \to 0$ and $r \to \infty$ we get

$$\int_{-\infty}^{\infty} \frac{x^4 dx}{x^6 - 1} + \frac{\pi i}{6} - \frac{\pi i}{6} = \frac{\pi\sqrt{3}}{3}$$

$$\therefore 2\int_{0}^{\infty} \frac{x^4 dx}{x^6 - 1} = \frac{\pi\sqrt{3}}{3}$$

$$\therefore \int_{0}^{\infty} \frac{x^4 dx}{x^6 - 1} = \frac{\pi\sqrt{3}}{6}.$$

Type 3

$\int_{-\infty}^{\infty} \frac{g(x)}{h(x)} \cos ax\, dx$ or $\int_{-\infty}^{\infty} \frac{g(x)}{h(x)} \sin ax\, dx$ where $g(x)$ and $h(x)$ are real polynomials such that degree of $h(x)$ exceeds that of $g(x)$ by at least one and $a > 0$.

Case (i) h(x) has no zeros on the real axis.

In this case take $f(z) = \frac{g(z)}{h(z)} e^{iaz}$

Therefore $f(z)$ has no poles on the real axis.
Choose the contour as in type 2 and proceeding as in type 2, we get the value of $\int_{-\infty}^{\infty} \frac{g(x)}{h(x)} e^{iax}$ dx

Taking the real and imaginary parts of $\frac{g(x)}{h(x)} e^{iax}$ dx, we obtain the values of $\int_{-\infty}^{\infty} \frac{g(x)}{h(x)} \cos ax$ dx and $\int_{-\infty}^{\infty} \frac{g(x)}{h(x)} \sin ax$ dx

Case (ii) $h(x)$ has zeros of order one on the real axis.

Take $f(z) = \frac{g(z)}{h(z)} e^{iaz}$. We notice that $f(z)$ has real poles.

Suppose a is a real zero of $h(x)$ on the real axis. In this case we indent the real axis as Case(ii) of Type 2 and evaluate the integral. To prove that the integral over the upper semicircle tends to zero as $r \to \infty$, we use the following lemma.

Jordan's Lemma 10.2.12

Let $f(z)$ be a function of the complex variables z satisfying the following conditions.

(i) $f(z)$ is analytic in upper half plane except at a finite number of poles.

(ii) $f(z) \to 0$ uniformly as $|z| \to \infty$ with $0 \le \arg z \le \pi$.

Space for Hints

(iii) a is a positive integer.
Then

$\lim\limits_{r \to \infty} \int\limits_C f(z) e^{iaz} dz = 0$ where C is the semi circle with centre at the origin and radius r.

Solved Problems

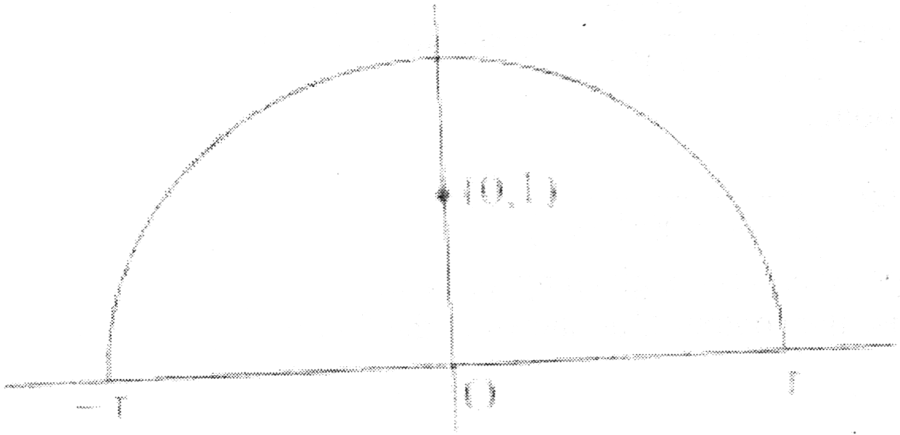
Problem 10.2.13

Prove that $\int\limits_0^\infty \frac{\cos x}{1+x^2} dx = \frac{\pi}{2e}$

Solution.

Let $f(z) = \frac{e^{iz}}{1+z^2}$.

The poles of $f(z)$ are given by i and $-i$.
Choose the contour C as shown in the figure.



The pole of $f(z)$ that lies within C is i. Hence by residue theorem ,

$$\int\limits_C f(z)dz = 2\pi i \, Res\, \{f(z)\, ;\, i\}$$

$$= 2\pi i \frac{h(i)}{k'(i)} \quad \text{where } h(z) = e^{iz} \text{ and } k(z) = 1 + z^2$$

$$= \frac{2\pi i e^{-1}}{2i} = \frac{\pi}{e}$$

$$\int\limits_{-r}^{r} \frac{e^{iax}}{1+x^2} dx + \int\limits_{C_1} \frac{e^{iaz}}{1+z^2} dz = \frac{\pi}{e}$$

When $r \to \infty$, the integral over C_1 tends to zero.

$$\int\limits_{-\infty}^{\infty} \frac{e^{iax}}{1+x^2} dx = \frac{\pi}{e}$$

Space for Hints

Equation real parts, we get, $\int_{-\infty}^{\infty} \frac{\cos x}{1+x^2} dx = \frac{\pi}{e}$.

$\therefore 2\int_{0}^{\infty} \frac{\cos x}{1+x^2} dx = \frac{\pi}{e}$ (since $\frac{\cos x}{1+x^2}$ is an even function.)

$\therefore \int_{0}^{\infty} \frac{\cos x}{1+x^2} dx = \frac{\pi}{2e}$.

Problem 10.2.14

Using the method of contour integration

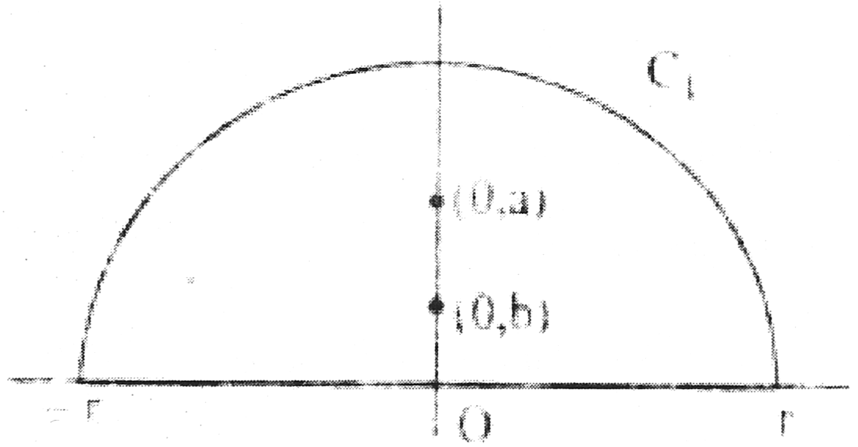
evaluate $\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+a^2)(x^2+b^2)} dx$,(a > b > 0).

Solution.

Let $f(z) = \frac{e^{iz}}{(z^2+a^2)(z^2+b^2)}$.

The poles of $f(z)$ are given by $ia, -ia, ib, -ib$.

Choose the contour C as shown in the figure.



The poles of $f(z)$ that lies within C are ia and ib.

Hence by residue theorem

$\int_C f(z)dz = 2\pi i$ (sum of the residures of $f(z)$).--------------------(1)

We find the residues of $f(z)$.

$Res\ \{f(z);\ a_i\} = \frac{h(ai)}{k'(ai)}$ where $h(z) = e^{iz}$ and $k(z) = (z^2+a^2)(z^2+b^2)$

$= z^4 + (a^2+b^2)z^2 + a^2b^2$ so that $4z^3 + 2(a^2+b^2)z$.

Space for Hints

$$\therefore \operatorname{Res}\{f(z);ai\} = \frac{e^{-a}}{4(ia)^3 + 2(a^2+b^2)(ia)}$$

$$= \frac{e^{-a}}{i\,2a[(a^2+b^2)-2a^2]}$$

$$= \frac{-ie^{-a}}{2a(b^2-a^2)}$$

$$= \frac{ie^{-a}}{2a(a^2-b^2)}.$$

Similarly, $Res\,\{f(z);\, b_i\} = \dfrac{ie^{-b}}{2b(b^2-a^2)}$

$$= \frac{-ie^{-b}}{2b(a^2-b^2)}$$

From (1), $\displaystyle\int_C f(z)dz = 2\pi i\left[\frac{i}{2(a^2-b^2)}\left(\frac{e^{-a}}{a}-\frac{e^{-b}}{b}\right)\right]$

$$= \frac{-\pi}{a^2-b^2}\left(\frac{e^{-a}}{a}-\frac{e^{-b}}{b}\right)$$

$$= \frac{\pi}{a^2-b^2}\left(\frac{e^{-b}}{b}-\frac{e^{-a}}{a}\right) \text{-------------------(2)}$$

Also (1) can be written using (2) as

$$\int_{C_1} f(z)dz + \int_{-r}^{r} \frac{e^{ix}}{(x^2+a^2)(x^2+b^2)}dx = \frac{\pi}{a^2-b^2}\left(\frac{e^{-b}}{b}-\frac{e^{-a}}{a}\right) \text{------(3)}$$

Further the integral over C_l tends to 0 as $r \to \infty$.
Therefore (3) becomes

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{(x^2+a^2)(x^2+b^2)}dx = \frac{\pi}{a^2-b^2}\left(\frac{e^{-b}}{b}-\frac{e^{-a}}{a}\right).$$

Equating real parts on both sides we get

$$\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+a^2)(x^2+b^2)}dx = \frac{\pi}{a^2-b^2}\left(\frac{e^{-b}}{b}-\frac{e^{-a}}{a}\right).$$

Problem 10.2.15

Prove that $\displaystyle\int_0^{\infty} \frac{\cos ax}{(1+x^2)^2}dx = \frac{\pi}{4}(a+1)e^{-a}$ where $a > 0$.

Solution.

Space for Hints

Let $f(z) = \dfrac{e^{iaz}}{(1+z^2)^2}$.

The poles of $f(z)$ are given by i and $-i$ which are double poles. Now choose the contour as in problem 10.2.13. The pole of $f(z)$ that lies within C is i.

$$\therefore \text{Res}\{f(z);i\} = \frac{1}{1!}g'(i) \text{ where } g(z) = (z-i)^2 f(z) = \frac{e^{iaz}}{(i+z)^2}$$

Therefore, $g'(z) = \dfrac{(z+i)^2 iae^{iaz} - e^{iaz}2(z+i)}{(z+i)^4}$

$$\therefore \text{Res}\{f(z);i\} = \frac{-4iae^{-a} - e^{-a}(4i)}{16} = \frac{-ie^{-a}(a+1)}{4}.$$

Hence by Cauchy's residue theorem,

$$\int_C f(z)dz = 2\pi i\left(\frac{ie^{-a}(a+1)}{4}\right) = \frac{\pi(a+1)e^{-a}}{2}$$

$$\therefore \int_{C_1} f(z)dz + \int_{-r}^{r} f(x)dx = \frac{\pi(a+1)e^{-a}}{2}$$

As $r \to \infty$, the integral over C_1 tends to zero.

$$\int_{-\infty}^{\infty} f(x)dx = \frac{\pi(a+1)e^{-a}}{2}$$

Equating real parts $\displaystyle\int_0^{\infty} \frac{\cos ax}{(1+x^2)^2}dx = \frac{\pi}{2}(a+1)e^{-a}$

Therefore, $\displaystyle\int_0^{\infty} \frac{\cos ax}{(1+x^2)^2}dx = \frac{\pi}{4}(a+1)e^{-a}$.

Problem 10.2.16

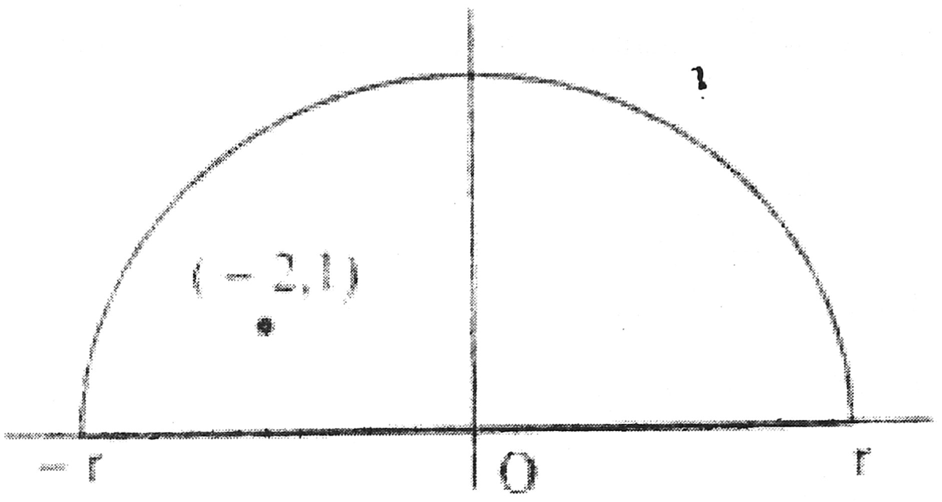
Prove that $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x}{x^2+4x+5}dx = -\frac{\pi \sin 2}{e}$

Solution.

Let $f(z) = \dfrac{e^{iz}}{z^2+4z+5}$.

The poles of $f(z)$ are the roots of the equation $z^2+4z+5=0$. They are given by $z = \dfrac{-4 \pm \sqrt{16-20}}{2} = -2 \pm i$

Choose the contour C as shown in the figure.



Space for Hints

$-2+i$ *is* only the pole of *f(z)* that lies within *C* and it is a simple pole. Hence by residue theorem,

$$\int_C f(z)dz = 2\pi\, i Res\,\{f(z)\,;-2+i\}$$

$$= 2\pi\, i\frac{h(-2+i)}{k'(-2+i)} \text{ where } h(z) = e^{iz} \text{ and } k(z) = z^2 + 4z + 5$$

$$\therefore \int_{C_1} f(z)dz + \int_{-r}^{r} f(x)dx = \frac{\pi e^{-2i}}{e}$$

Since the integral over C_1 tends to zero as $r \to \infty$, we have

$$\int_{-\infty}^{\infty} f(x)dx = \frac{\pi e^{-2i}}{e} = \frac{\pi}{e}(\cos 2 - i\sin 2).$$

Equation imaginary parts we get

$$\int_{-\infty}^{\infty} \frac{\sin x}{x^2+4x+5}dx = -\frac{\pi \sin 2}{e}.$$

Problem 10.2.17

Evaluate $\int_0^{\infty} \frac{x\sin x}{x^2+a^2}dx$

Solution.

Let $f(z) = \dfrac{ze^{iz}}{(z^2+a^2)}$.

The poles of *f(z)* are given by *ia* and *–ia* which are simple poles. Now choose the contour as in problem 10.2.13. Only the pole $z = ia$ lies inside *C*

Space for Hints

$\operatorname{Res}\{f(z);ia\} = \dfrac{h(ia)}{k'(ia)}$ where $h(z)=ze^{iz}$ and $k(z) = z^2 + a^2$ so that $k'(z)= 2z$.

$$\operatorname{Res}\{f(z);ia\} = \frac{iae^{i(ia)}}{2(ia)} = \frac{e^{-a}}{2}$$

Hence by Cauchy's residue theorem,

$$\int_C f(z)dz = 2\pi i\left(\frac{e^{-a}}{2}\right) = \pi i e^{-a}$$

$$\therefore \int_{C_1} f(z)dz + \int_{-r}^{r} f(x)dx = \pi i e^{-a}.$$

As $r\to\infty$, the integral over C_1 tends to zero.

$$\therefore \int_{-\infty}^{\infty} f(x)dx = \pi i e^{-a}.$$

$$\therefore \int_{-\infty}^{\infty} \frac{xe^{ix}}{x^2 + a^2}dx = \pi i e^{-a}.$$

$$\therefore \int_{-\infty}^{\infty} \frac{x(\cos x + i\sin x)dx}{x^2 + a^2}dx = \pi i e^{-a}.$$

Equating imaginary parts on both sides, we get,

$$\therefore \int_{-\infty}^{\infty} \frac{x\sin x dx}{x^2 + a^2}dx = \pi\ e^{-a}.$$

Since the above integrand is an even function, we have,

$$2\int_0^{\infty} \frac{x\sin x dx}{x^2 + a^2}dx = \pi\ e^{-a}.$$

$$\therefore \int_0^{\infty} \frac{x\sin x dx}{x^2 + a^2}dx = \frac{\pi e^{-a}}{2}.$$

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